A Unified Dynamic Model Formulation for Robotic Manipulator Systems

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This paper addresses the problem of formulation of a unified dynamic model for sundry robotic manipulator systems derived from the first principle of mechanics instead of the existing formulation based on linear separability principle. It provides a systematic derivation, evaluation and subsequent conceptual interpretation of manipulator dynamics model. Further, it analyzes the generality of the unified model over a wide range of manipulator configurations. In addition, it describes the implementation aspects of the unified model.

1. INTRODUCTION

Dynamics modeling serves as an essential component to study structure analysis, control algorithm development and system motion simulation for robot manipulators. The importance of manipulator dynamics modeling has been well covered by robotics researchers. Various applications of robotic manipulator systems show that a similar manipulator structure can be used in different applications with the subjugation of stringent environment constraints. These manipulators can be a combination of rigid and flexible links and joints. In addition, the manipulator base can be rigidly fixed on the earth or completely free (space manipulators) or flexible or partially motion constrained as in mobile robots on fixed surface. Further, multiple manipulators can work cooperatively or independently amidst obstacles or without obstacles. Also, one of the important issues is the type of joint. A joint can have zero degrees of free-

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dom (rigid contacts) or at best six degrees of freedom (completely free, i.e., physically not connected with any other bodies).[1–3] Deriving a unified dynamic model taking into consideration all these aspects may be a formidable task although not impossible. Further, it can be seen that the two extreme cases of base mobility are an immovable base with zero degrees of freedom and a completely free base with six degrees of freedom (space manipulators). A flexible base is very similar to a free-floating base with the exception that the free-floating base is a floating inertia whereas a flexible base may be either a flexible beam or a spring and damper (visco-elastic) system.[4] That means that an additional base force is needed to deflect the base in the case of a flexible base manipulator system. In addition, a mobile base can be modeled as a partially constrained system with degrees of freedom less than six.[5] Apart from this, a floating base can serve as a fixed (immovable) base when its mass and inertia are both infinite. Hence, a floating base with external forces can be considered as a basis for modeling any type of base.

Although, an extensive amount of research has been carried out on the kinematics and dynamics of terrestrial and space based single and multi-arm, fully-actuated and underactuated, rigid and flexible link, base and joint-based manipulator systems, the importance of developing a unified kinematics and dynamics model is not established. Keeping in view the generality of a complete free base that represents a space robot, all the discussion in this paper is concentrated on multi-arm rigid link space manipulator systems. The dynamics of a space manipulator system with completely free base is much more complex than that of its fixed base counterpart. To cope with the space manipulators, without loss of generality, the gravity term has been assumed to be zero, which in turn yields simple and lucid dynamic equations. The gravity terms can be very easily incorporated to the manipulator dynamics[6] whenever it is needed. Further, a space manipulator system with base free is nothing but an underactuated manipulator system. Therefore, the analysis presented here can very easily incorporate the underactuated systems such as flexible-base, flexible-arm and free-joint manipulator systems.[4] Further, incorporation of multiple arms working cooperatively to handle an object mounted on a free base can possibly be treated as a generalized manipulator system. Then, the problem of dynamics modeling of such cooperating multi-arm manipulator system can precisely be stated as below.

Problem Statement: Formulate a dynamic model for a multi-arm rigid link co-operative manipulator system mounted on a completely free base in the presence of external forces such that the model will be representative of sundry manipulator configurations subject to the following assumptions:

- Completely free base represents all varieties of possible bases starting from fixed to free and rigid to flexible.
- Variables of free base represent all possible passive joints in the system (which covers all under-actuated systems).
- There is rigid contact between the end-effector and the object in a multi-manipulator system co-operatively handling a common object.
- No collision when multiple manipulators working independent of each other.
- No obstacles in the environment.
- Effects of gravity are neglected.

Yoshida and Nenchev[4] presented a unified model based on the concepts originally proposed by Jain and Rodriguez.[7] Yoshida and Nenchev's model is based on the concept of decomposition of any manipulator system into one active and one passive manipulator system. Then, the respective independent behaviors were superimposed to represent back the complete system. This model is described in Appendix A. However, in this paper the unified model will be derived from the basic principles of mechanics of manipulation. Then, it will be shown that the inherent assumption of linear separability of an underactuated system into its active and passive subsystems to formulate the model may not represent the true model of the original manipulator system. In addition, this paper deals explicitly with the calculation of velocity and acceleration of each of the links and base, and torques acting on each of the joints with the knowledge of the forces acting at the end-effector terminals.

In Section 2, a vivid description of the entire manipulator system is presented. Subsequently, in Section 3, the basic spatial operators used in this work to derive all mathematical representations are discussed. The spatial operator algebra (SOA) is selected to represent manipulator kinematics and dynamics as it provides an in-depth physical insight into the high-level analytical manipulator motion expressions by representing those with easily manipulable form. Further, the unified dynamic model formulation utilizes standard Newton–Euler equations of motion, due to its obvious advantages.[8,9] Both kinematics and dynamics of a space manipulator system heavily depend on the mass and inertia of the common base and the rest of the system. The complexity of the interactions between the motion of the free-base and the rest

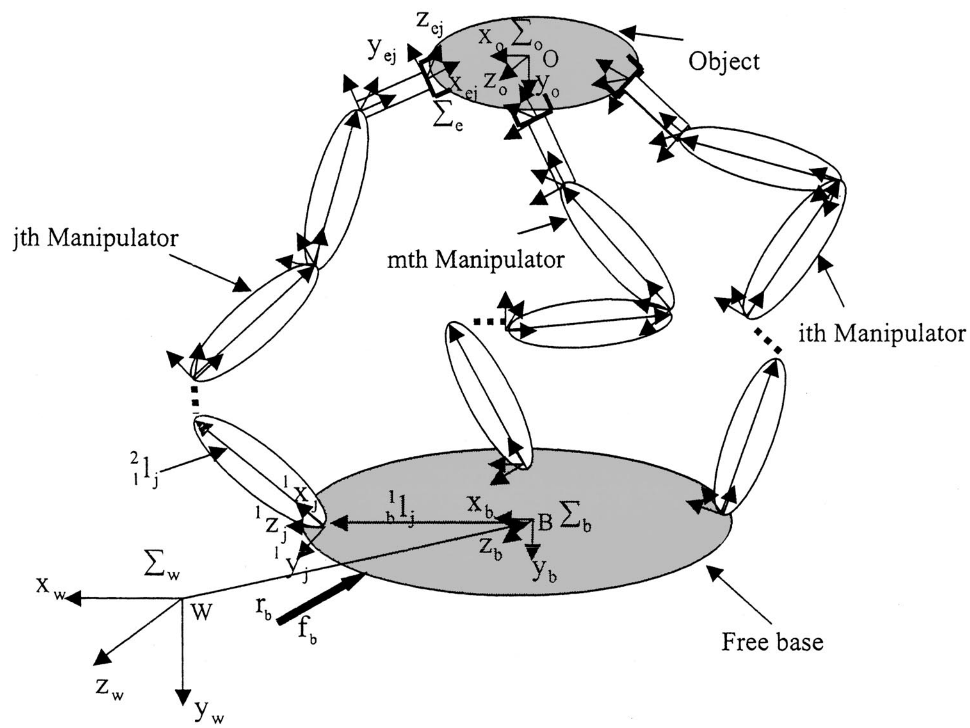

Figure 1. A multi-arm co-ordinating space robotic system.

of the system has been systematically studied in Sections 4 and 5. Explicit mathematical relationships among them have been presented.

A clear understanding of the mathematical representations of a complex system is only possible by proper physical interpretation of the mathematical terms or expressions. Section 6 deals with this issue by providing a physical interpretation of the equation of motion of the whole manipulator system.

Forward dynamics analysis is important for simulating a system. This requires the determination of the system acceleration that results from given current states of the system, and joint and base forces/torques. Then, this acceleration is integrated to calculate the next joint/base positions and velocities. This issue is discussed in Section 7.

In essence, the unified dynamic model derived here with reference to a space manipulator system, incorporates systems containing any number of manipulators with any number of rigid links connected by any type of joints and subject to known external forces. Thus, the model obtained is a completely generalized model for any manipulator system. The generality of the derived model has been shown explicitly in Section 8 by presenting the particular cases of this model to represent any underactuated manipulator system. Then, the computer implementation of the system of equations is discussed in Section 9. Subsequently, Section 10 discusses the importance of the derived model for sundry manipulator configurations. Finally, a conclusion of the works presented in this paper is drawn in Section 11.

2. MANIPULATOR SYSTEM DESCRIPTION

The general model of a multi-arm co-ordinating space manipulator system with m-robots handling a common object, which are then mounted on a completely free base, is shown in Figure 1. Each robot consists of n rigid bodies known as links. Each adjacent link is connected by means of a joint with multiple degrees of freedom. The ith joint of the jth manipulator connects the $(i\text{-}l)$th and ith rigid bodies together with degrees of freedom ${}^{i}d_j$, where $0<{}^{i}d_j<6$. Then, ${}^{0}d_j=d_b$ is the degrees of freedom of the base body; in other words, the zeroth joint connects the base body to the world reference frame widely known as inertial frame. Hence, the common base for the manipulators serves as the zeroth link of the entire system, and other links are numbered in increasing order from base to the end-effector. For a space manipulator system, the base is mobile with complete motion freedom, where the attitude can rotate about three axes as well as translate along spatial x, y and z axes, can be modeled as a six degree of freedom joint, i.e., $d_b = 6$. Whereas, for a terrestrial mobile manipulator system, this joint can be modeled as a partially constrained system with less than six degrees of freedom. This has been shown in Ziauddin,[5] where he has modeled a mobile two-arm co-operative manipulator system with three degrees of freedom joint between the base and inertial frame. Similarly, for a fixed base terrestrial robot $d_b = 0$. In general, the degrees of freedom of the entire manipulator system can be represented as

$$d_t = d_b + \sum_{j=1}^{m} d_j = d_b + d_m,$$

where $d_j = \Sigma_{i=1}^{n+1} d_j$ is the degrees of freedom of the jth manipulator, and d_m is the total degrees of freedom of all the manipulators.

Without loss of generality, we have assumed that only one external spatial force $\mathbf{f}_b$ is applied to the moving platform of the system. In the absence of any external force, the momentum of the whole system can be treated as a constant. In space operation, often to conserve energy, the spacecraft thruster is closed once the robotic system acquires the required position. This type of system is called *free-floating*.[10] On the other hand, in the case of a *free-flying* manipulator system, both the spacecraft and manipulators are controlled simultaneously. Now, for the free-flying robotic systems, the spacecraft thruster force required to control its position and attitude can be equivalently modeled as external force acting on the platform. Hence, in the case of a free-floating space robotic system, this external force can be assumed to be zero. Further, this concept of base external force can be extended for flexible base manipulator systems (FBMSs).[11,12] In FBMS, the base force is modeled as the sum of the damping and spring forces.

A space manipulator system possesses base-invariance symmetry,[13] which states that any of its constituent rigid bodies (links or spacecraft) can be chosen as a *base body* or *prime body* (PB).[14] In that paper, Saha showed that the end-effector serving as a PB results in computationally efficient kinematic equations if the end-effector motion is the only concern, otherwise it is essential to choose the moving platform or the base as the PB. As this research work is concerned with the development of a unified kinematic and dynamic model for various manipulator systems, it is necessary to choose the base as the PB.

3. SPATIAL NOTATION

In the last decade, the notion of spatial force, acceleration and inertia[15,16] have been studied extensively for robot manipulator system representation. Rodriguez[17] extended these concepts to formulate different operators that are used to solve manipulator modeling and control problems. These spatial operators are commonly known as *spatial operator algebra* (SOA). Later this concept was expounded and used extensively by Rodriguez *et al.*,[18] and Rodriguez and Kreutz.[19] SOA serves as a very good tool for manipulator modeling and control. This provides a framework for clearly understanding the dynamics of the systems of rigid bodies interacting among themselves and their environment. When operated on velocities and accelerations, these spatial operators always yield close form dynamic equations of motion that arise from Lagrangian analysis.[18] SOA primarily offers a mathematical framework, and its simplicity is its potential to deal with complex manipulator dynamics analysis, advanced control and motion planning problems.[20] These advantages of SOA have been exploited in this work to present a generalized dynamic model for robotic manipulators.

The role of SOA for single arm robotic manipulators was established in Rodriguez.[17] Later, Rodriguez *et al.*,[18] Rodriguez and Kreutz[19] and Kreutz *et al.*[20] carried out extensive work to show the potential of SOA for robot modeling and control. The effects of spatial operators to yield very simple statement and solution of the forward dynamics of multiple arms manipulating a common object have been presented in Rodriguez.[21] Jain and Rodriguez[7] applied SOA to model underactuated manipulators with special emphasis on space manipulators. This served as a basis for formulating the unified model presented in Yoshida and Nenchev.[4]

The spatial operator algebra (SOA) framework described above is a co-ordinate free notation, where the co-ordinate frames are defined according to a modified form of the Denavit–Hartenberg convention. As per this convention, the co-ordinate frame of a particular link is attached to that link with frame origin at the near end of the link. The co-ordinate frames are shown in Figure 1, and those are assigned to the world reference/inertial (Σ_w), base (Σ_b), end-effectors (Σ_e) and object frames (Σ_o) with (x_w, y_w, z_w), (x_b, y_b, z_b), $\{(x_{ci}, y_{cj}, z_{cj}),\ j = 1,2,....,m\}$, and (x_o, y_o, z_o), respectively. Further, O and B are the center of mass of object and base, respectively. In addition, W serves as the origin of the world reference frame.

The spatial velocity, acceleration and force vectors of the ith link of the jth robot resolved in the ith link frame are denoted by the vector symbols ${}^i\mathbf{V}_j$, ${}^i\dot{\mathbf{V}}_j$ and ${}^i\mathbf{f}_j$, respectively, and are defined as ${}^i\mathbf{V}_j = [{}^i\omega_j^T \, {}^iv_j^T]^T \in \mathbb{R}^6$, where ${}^i\omega_j \in \mathbb{R}^3$ and ${}^iv_j \in \mathbb{R}^3$ are the angular and linear velocity vectors, respectively, of the ith link of the jth robot. Here, $\mathbb{R}^n$ is the n-dimensional Eucledian space. ${}^i\dot{\mathbf{V}}_j = [{}^i\dot{\omega}_j^T \, {}^i\dot{v}_j^T]^T \in \mathbb{R}^6$, where ${}^i\dot{\omega}_j \in \mathbb{R}^3$ and ${}^i\dot{v}_j \in \mathbb{R}^3$ are the angular and linear acceleration vectors, respectively, of the ith link of the jth robot. ${}^i\mathbf{f}_j = [{}^i\eta_j^T \, {}^if_j^T]^T \in \mathbb{R}^6$, where ${}^if_j \in \mathbb{R}^3$ and ${}^i\eta_j \in \mathbb{R}^3$ are the force and moment vectors, respectively of the ith link of the jth robot at the ith frame origin.

The spatial transformation matrix ${}^{\;\;i}_{i-1}\mathbf{X}_j$ transforms a spatial velocity from the $(i-1)$th co-ordinate frame to the ith co-ordinate frame of the jth robot and is defined as

$${}^{\;\;i}_{i-1}\mathbf{X}_j = \begin{bmatrix} {}^{\;\;i}_{i-1}\mathbf{R}_j & \mathbf{0} \\ {}^{\;\;i}_{i-1}\mathbf{R}_j \, {}^{\;\;i}_{i-1}\tilde{\mathbf{p}}_j^T & {}^{\;\;i}_{i-1}\mathbf{R}_j \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

where ${}^{\;\;i}_{i-1}\mathbf{R}_j \in \mathbb{R}^{3\times 3}$ is a rotation matrix from the $(i-1)$th link frame to the ith link frame for the jth robot; $\mathbb{R}^{m\times n}$ is the matrix of order $m \times n$; ${}^{\;\;i}_{i-1}\mathbf{p}_j \in \mathbb{R}^3$ is a vector from the origin of the $(i-1)$th link frame to the origin of the ith link frame for the jth robot; $\tilde{\mathbf{p}} \in \mathbb{R}^{3\times 3}$ for a vector $\mathbf{p} = [p_x \, p_y \, p_z]^T \in \mathbb{R}^3$ is an anti-symmetric matrix defined as

$$\tilde{\mathbf{p}} = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}.$$

Similarly, transformation matrix that transforms a spatial force from ith link frame to the $(i-1)$th link frame for the jth robot can be represented as ${}^{i-1}_{\;\;\;i}\mathbf{X}_j = {}^{\;\;i}_{i-1}\mathbf{X}_j^T \in \mathbb{R}^{6\times 6}$ and is defined as

$${}^{i-1}_{\;\;\;i}\mathbf{X}_j = \begin{bmatrix} {}^{i-1}_{\;\;\;i}\mathbf{R}_j & {}^{\;\;i}_{i-1}\tilde{\mathbf{p}}_j \, {}^{i-1}_{\;\;\;i}\mathbf{R}_j \\ \mathbf{0} & {}^{i-1}_{\;\;\;i}\mathbf{R}_j \end{bmatrix}.$$

The spatial inertia ${}^i\mathbf{M}_j$ of the ith link of the jth robot is defined as

$${}^i\mathbf{M}_j = \begin{bmatrix} {}^i\mathbf{I}_j & {}^im_j \, {}^c_i\tilde{\mathbf{l}}_j \\ -{}^im_j \, {}^c_i\tilde{\mathbf{l}}_j & {}^im_j\mathbf{E}_3 \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

where ${}^im_j$, the mass of the ith link of the jth robot; ${}^i\mathbf{I}_j \in \mathbb{R}^{3\times 3}$, the inertia tensor of the ith link of the jth robot at the ith frame origin; ${}^c_i\mathbf{l}_j \in \mathbb{R}^3$, the position vector from the ith frame origin to the center of mass of the ith link of the jth robot; and $\mathbf{E}_n \in \mathbb{R}^{n\times n}$, an identity matrix.

The general joint model used in this paper is described in Brandl *et al.*,[1] Roberson and Schwertassek,[2] and Lilly.[3] This general joint model has been defined with the incorporation of the orthogonal vectors ${}^i\mathbf{\Phi}_j$ and ${}^i\mathbf{\Phi}_j^c$, which represent matrices of free and constrained mode vectors of the ith joint of the jth robot, respectively.

4. KINEMATICS

In this section, the kinematics of a rigid multi-arm space manipulator system is developed using body fixed geometric vectors similar to the efficient direct path method.[22] The motion of the center of mass of the base decides the overall motion of the entire system with respect to the inertial frame. In the following subsections the position and velocity analysis, and the various Jacobian matrices associated with the entire manipulator system, are presented. The free body diagram (FBD) of the system without the object and base is shown in Figure 2.

4.1. Position Analysis

The inertial position of an arbitrary point p on the ith link of the jth robot represented by the position vector ${}^i\mathbf{r}_j^p \in \mathbb{R}^3$ and can be expressed as

$${}^i\mathbf{r}_j^p = \mathbf{r}_b + {}^i_b\mathbf{r}_j^p, \tag{1}$$

where ${}^i_b\mathbf{r}_j^p \in \mathbb{R}^3$ is the position vector of the point p with respect to the center of mass (CM) of the base and $\mathbf{r}_b \in \mathbb{R}^3$ is the inertial position of the base CM. Here, ${}^i_b\mathbf{r}_j^p$ can be expressed in terms of the link lengths as

$${}^i_b\mathbf{r}_j^p = {}^1_b\mathbf{l}_j + \sum_{k=1}^{i} {}^{k+1}_{\;\;\;k}\mathbf{l}_j + {}^p_i\mathbf{l}_j, \tag{2}$$

where ${}^1_b\mathbf{l}_j \in \mathbb{R}^3$ is the position vector for the joint 1 of the jth robot with respect to the center of mass of the platform, ${}^{i+1}_{\;\;\;i}\mathbf{l}_j \in \mathbb{R}^3$ is the length of the link connecting link frames i and $i+1$ of the jth robot, and ${}^p_i\mathbf{l}_j \in \mathbb{R}^3$ is the position vector of the point p with respect to the ith link frame origin of the jth robot.

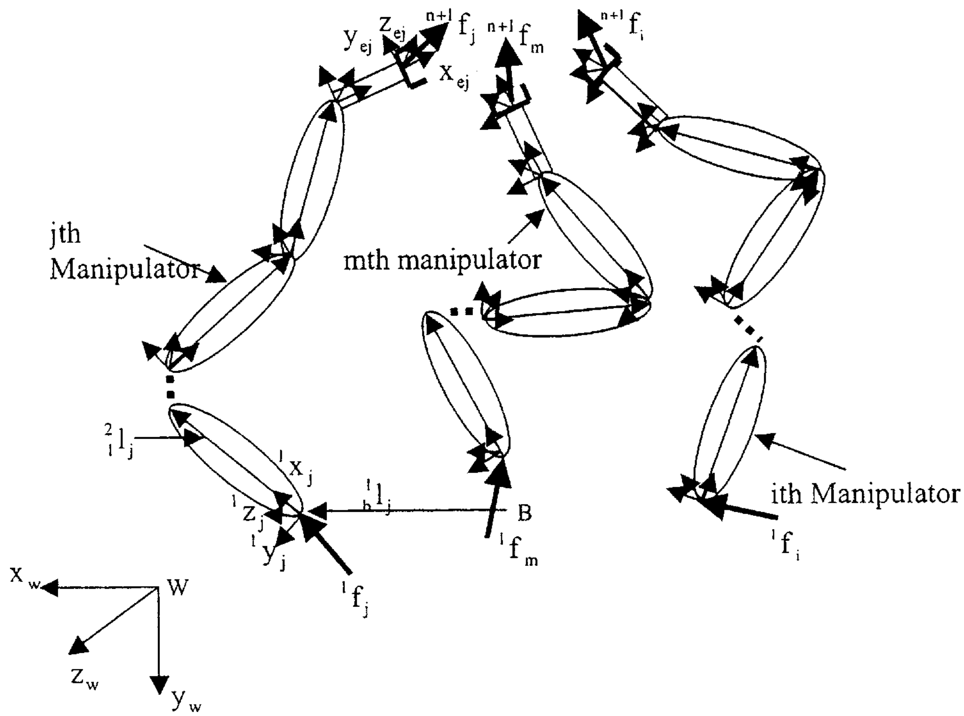

Figure 2. Free body diagram of the multi-arm robotic system without base and object.

Now, substitution of Eq. (2) in Eq. (1) yields

$$ {}^i\mathbf{r}_j^p = \mathbf{r}_b + {}^1_b\mathbf{l}_j + \sum_{k=1}^{i-1} {}^{k+1}_{k}\mathbf{l}_j + {}^p_i\mathbf{l}_j . \tag{3} $$

If the point p is located on the base of the manipulator system, then the position vector in Eq. (1) can be expressed as

$$ \mathbf{r}_b^p = \mathbf{r}_b + {}^p_b\mathbf{r}, \tag{4} $$

where $\mathbf{r}_b^p \in \mathbb{R}^3$ is the position vector of point p with respect to the base CM and ${}^p_b\mathbf{r} \in \mathbb{R}^3$ is the position vector of any point on the base.

4.2. Velocity Analysis

In this subsection, mathematical expressions for the velocity of all the links and the end-effectors will be formulated in terms of system Jacobians, which consist of different mechanical parameters of the manipulator system.

The inertial velocity of the point p defined in Eq. (3) can be obtained by differentiating this equation with respect to time, which yields

$$ {}^i\dot{\mathbf{r}}_j^p = \dot{\mathbf{r}}_b^w + \omega_b \times {}^1_b\mathbf{l}_j + \sum_{k=1}^{i-1} {}^k\omega_j \times {}^{k+1}_{k}\mathbf{l}_j + {}^k\omega_j \times {}^p_i\mathbf{l}_j , \tag{5} $$

where $\dot{\mathbf{r}}_b^w$ is the linear velocity of the base CM with respect to the world reference frame, and ${}^k\omega_j$ and ω_b are the angular velocities of the kth link of the jth robot and base, respectively. This angular velocity for single degree of freedom rotational joints between two links can be represented as

$$ {}^k\omega_j = \omega_b + \sum_{k=1}^{i} {}^k\dot{q}_j \, {}^k\hat{z}_j , \tag{6} $$

where ${}^k\dot{q}_j$ is the kth joint angle rate of the jth robot, and ${}^k\hat{z}_j$ is the unit vector along the axis of the kth joint of the jth robot. However, the representation of angular velocity for multiple degrees of freedom joints is discussed afterwards using SOA conventions.

The velocities in Eqs. (5) and (6) can easily be represented in the spatial operator notation discussed earlier. However, all these formulations are represented in the inertial co-ordinate frame, which is not computationally as efficient as representing these in the link co-ordinates.[16] Henceforth, all the mechanical quantities (velocity, acceleration, force, inertia etc.), of which only velocity is considered till now, will be assumed to be represented in the corresponding link coordinates, unless otherwise stated. Then, the spatial velocity of the ith link of the jth robot, corresponding to Eq. (5) as represented in the respective link frames, can be expressed as

$$^{i}\mathbf{V}_j = {}_{i-1}^{\;\;i}\mathbf{X}_j \, {}^{i-1}\mathbf{V}_j + {}^{i}\mathbf{\Phi}_j \, {}^{i}\dot{\mathbf{q}}_j \,, \tag{7}$$

where $^{i}\mathbf{V}_j \in \mathbb{R}^6$ is the velocity of the ith link of the jth robot, $^{i}\mathbf{\Phi}_j \in \mathbb{R}^{6\times {}^{i}d_j}$ is the free mode vector for the ith joint of the jth robot, and $^{i}\dot{\mathbf{q}}_j \in \mathbb{R}^{\,{}^{i}d_j}$ is the ith joint velocity of the jth robot.

Then, the spatial velocity $^{i}\mathbf{V}_j$ in Eq. (7) can further be expressed in terms of the base velocity as

$$^{i}\mathbf{V}_j = {}_{b}^{i}\mathbf{X}_j \mathbf{V}_b + \sum_{k=1}^{i} {}_{k}^{i}\mathbf{X}_j \, {}^{k}\mathbf{\Phi}_j \, {}^{k}\dot{\mathbf{q}}_j \,, \tag{8}$$

where $\mathbf{V}_b = [\omega_b^T \, \dot{\mathbf{r}}_b^T]^T \in \mathbb{R}^6$ is the base spatial velocity represented in the base frame with $\dot{\mathbf{r}}_b$ representing its linear velocity, and ${}_{b}^{i}\mathbf{X}_j \in \mathbb{R}^{6\times 6}$ is the transformation matrix between base and ith link frame of the jth robot. Here, the inherent properties of transformation matrices such as ${}_{i}^{i}\mathbf{X} = \mathbf{E}_6$ and ${}_{i}^{j}\mathbf{X} = {}_{k}^{j}\mathbf{X} \, {}_{i}^{k}\mathbf{X}$, $\forall (i,k)$ are used to obtain this relationship.

The spatial velocity of all the links of the jth robot can be expressed concisely from Eq. (7) as

$$\mathbf{V}_j = {}_{b}\mathbf{X}_j \mathbf{V}_b + \mathbf{X}_j \mathbf{\Phi}_j \dot{\mathbf{q}}_j \,, \tag{9}$$

where $\mathbf{V}_j = [{}^{1}\mathbf{V}_j^T \, {}^{2}\mathbf{V}_j^T \cdots {}^{n}\mathbf{V}_j^T]^T \in \mathbb{R}^{6n}$, $\mathbf{\Phi}_j = \mathrm{diag}({}^{1}\mathbf{\Phi}_j \, {}^{2}\mathbf{\Phi}_j \cdots {}^{n}\mathbf{\Phi}_j) \in \mathbb{R}^{6n\times d_j}$, $\dot{\mathbf{q}}_j = [{}^{1}\dot{\mathbf{q}}_j^T \, {}^{2}\dot{\mathbf{q}}_j^T \cdots {}^{n}\dot{\mathbf{q}}_j^T]^T \in \mathbb{R}^{d_j}$, ${}_{b}\mathbf{X}_j = [{}_{b}^{1}\mathbf{X}_j^T \, {}_{b}^{2}\mathbf{X}_j^T \cdots {}_{b}^{n}\mathbf{X}_j^T]^T \in \mathbb{R}^{6n\times 6}$, and the matrix $\mathbf{X}_j$ is given as

$$\mathbf{X}_j = \begin{bmatrix} {}_{1}^{1}\mathbf{X}_j & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ {}_{1}^{2}\mathbf{X}_j & {}_{2}^{2}\mathbf{X}_j & \mathbf{0} & \cdots & \mathbf{0} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ {}_{1}^{n}\mathbf{X}_j & {}_{2}^{n}\mathbf{X}_j & \cdot & \cdots & {}_{n}^{n}\mathbf{X}_j \end{bmatrix} \in \mathbb{R}^{6n\times 6n}. \tag{10}$$

For all the m robots, Eq. (9) can be concisely expressed as

$$\mathbf{V} = \mathbf{X}_b \mathbf{V}_b + \mathbf{X}\mathbf{\Phi}\dot{\mathbf{q}}, \tag{11}$$

where $\mathbf{V} = [\mathbf{V}_1^T \mathbf{V}_2^T \cdots \mathbf{V}_m^T]^T \in \mathbb{R}^{6nm}$, $\mathbf{X}_b = [{}_{b}\mathbf{X}_1^T \, {}_{b}\mathbf{X}_2^T \cdots {}_{b}\mathbf{X}_m^T]^T \in \mathbb{R}^{6nm\times 6}$, $\mathbf{X} = \mathrm{diag}(\mathbf{X}_1 \mathbf{X}_2 \cdots \mathbf{X}_m) \in \mathbb{R}^{6nm\times 6nm}$, $\mathbf{\Phi} = \mathrm{diag}(\mathbf{\Phi}_1 \mathbf{\Phi}_2 \cdots \mathbf{\Phi}_m) \in \mathbb{R}^{6nm\times d_m}$, and $\dot{\mathbf{q}} = [\dot{\mathbf{q}}_1^T \dot{\mathbf{q}}_2^T \cdots \dot{\mathbf{q}}_m^T]^T \in \mathbb{R}^{d_m}$.

Interpretation of the spatial velocity vector for the entire manipulator-base system expressed in Eq. (11) is simplified if it is expressed in terms of its individual component matrices. The block diagonal matrix operator $\mathbf{\Phi}$, when it acts on $\dot{\mathbf{q}}$, results in a vector of relative spatial link velocities. Due to its block diagonal feature, it is *memoryless* or nonrecursive. Then, $\mathbf{X}$ acts on $\mathbf{\Phi}\dot{\mathbf{q}}$ to provide the composite vector of link spatial velocities. Similarly, $\mathbf{X}_b$ projects the base velocity $\mathbf{V}_b$ onto the respective joint frames. Hence, $\mathbf{X}_b$ and $\mathbf{X}$ represent the propagation of the base and joint velocities, respectively, across the link frames. It is noteworthy that $\mathbf{X}$ is lower block triangular, hence is *causal* in nature. This term will be explained in detail in Section 5 during the discussion of force propagation.

The end-effector velocity of the jth robot, denoted by $\mathbf{V}_j^e \in \mathbb{R}^6$, is defined as per Eq. (8) as

$$\mathbf{V}_j^e = {}^{n+1}\mathbf{V}_j = {}_{\;n}^{n+1}\mathbf{X}_j \, {}^{n}\mathbf{V}_j \,. \tag{12}$$

Here, $(n+1)$th frame is the end-effector frame, and ${}_{\;n}^{n+1}\mathbf{X}_j$ maps the spatial velocity of the nth link frame to $(n+1)$th or the end-effector frame.

Now using Eq. (8), we can get

$$\begin{aligned} \mathbf{V}_j^e &= {}_{\;n}^{n+1}\mathbf{X}_j \left[{}_{b}^{n}\mathbf{X}_j \mathbf{V}_b + \sum_{k=1}^{n} {}_{k}^{n}\mathbf{X}_j \, {}^{k}\mathbf{\Phi}_j \, {}^{k}\dot{\mathbf{q}}_j \right] \\ &= \mathbf{B}_j \, {}_{b}\mathbf{X}_j \mathbf{V}_b + \mathbf{B}_j \mathbf{X}_j \mathbf{\Phi}_j \dot{\mathbf{q}}_j \,, \end{aligned} \tag{13}$$

where $\mathbf{B}_j = [\mathbf{0} \cdots \mathbf{0} \; {}_{\;n}^{n+1}\mathbf{X}_j] \in \mathbb{R}^{6\times 6n}$.

In Eq. (12), $\mathbf{B}_j$ acts on link spatial velocities $\mathbf{X}_j \mathbf{\Phi}_j \dot{\mathbf{q}}_j$ to propagate the individual link frame spatial velocities to form the end-effector velocities. The total end-effector velocity is the sum of the transformed base velocity and link velocities to the end-effector frame.

The end-effector velocity due to all the manipulators yields

$$\mathbf{V}^e = \mathbf{B}\mathbf{X}_b \mathbf{V}_b + \mathbf{B}\mathbf{X}\mathbf{\Phi}\dot{\mathbf{q}} = \mathbf{J}_b \mathbf{V}_b + \mathbf{J}_q \dot{\mathbf{q}}, \tag{14}$$

where $\mathbf{J}_b \in \mathbb{R}^{6m\times 6}$ is the base Jacobian, $\mathbf{J}_q \in \mathbb{R}^{6m\times d_m}$ is the link Jacobian and $\mathbf{B} = \mathrm{diag}(\mathbf{B}_1 \mathbf{B}_2 \cdots \mathbf{B}_m) \in \mathbb{R}^{6m\times 6nm}$. Hence, $\mathbf{J}_b = \mathbf{B}\mathbf{X}_b$ and $\mathbf{J}_q = \mathbf{B}\mathbf{X}\mathbf{\Phi}$.

In Eq. (14) $\mathbf{J}_b$ is the base Jacobian that represents the contribution of the base velocity on the end-effector velocity, whereas $\mathbf{J}_q$ is the link Jacobian matrix that describes the motion induced at the end-effector due to the motion of the active degrees of freedom. The action of $\mathbf{J}_q$ can be summarized as (*i*) $\mathbf{\Phi}\dot{\mathbf{q}}$ results in relative spatial velocities between the links along the joint axes; (*ii*) **X** then causally propagates these relative velocities starting from the base to the tip to form all the link spatial velocities; and (*iii*) **B** then projects out the last link spatial velocities ${}^{n}\mathbf{V}_j$, $j=1,...,m$ from the link spatial velocity vector and propagates it to the tip forming $\mathbf{V}^e$. Then, the action of $\mathbf{J}_b$ can be summarized as (*i*) $\mathbf{X}_b$ projects base velocity $\mathbf{V}_b$ on to all the active joints; (*ii*) then **B** projects the effects of $\mathbf{V}_b$ on *n*th link frame onto the end-effector frame.

Both base and link Jacobians $\mathbf{J}_b$ and $\mathbf{J}_q$ are independent of dynamical quantities such as link masses and inertia. They depend only upon their kinematical properties.

5. INVERSE DYNAMICS

The inverse dynamics problem is concerned with calculating the driving forces/torques needed to produce a prescribed motion with a given current state joint/base position and velocity, and desired acceleration. The inverse dynamics problem serves as the foremost problem to be solved for the control of any manipulator system. This vital issue of a robotic manipulator system is addressed in the following subsections.

5.1. Force Relations

The objective of this subsection is to formulate mathematical expressions for the forces acting on different parts of the manipulator system in terms of the link and base motion variables, and manipulator system parameters. Particularly, the effects of external forces acting at the end-effectors on the links and base of the manipulator system will be represented mathematically.

It has been shown that the acceleration of the base $\dot{\mathbf{V}}_b$ is dependent on the motion of the manipulators, and so can only be obtained after the computation of the manipulator motion variables. Now, the acceleration of the links of the manipulators can be obtained by differentiating Eq. (11) with respect to time, which yields

$$\begin{aligned}\dot{\mathbf{V}} &= \dot{\mathbf{X}}_b\mathbf{V}_b + \mathbf{X}_b\dot{\mathbf{V}}_b + \dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}} + \mathbf{X}\boldsymbol{\Phi}\ddot{\mathbf{q}} \\ &= \mathbf{X}\boldsymbol{\Phi}\ddot{\mathbf{q}} + \dot{\mathbf{X}}_b\mathbf{V}_b + \mathbf{X}_b\dot{\mathbf{V}}_b + \dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}}, \end{aligned} \tag{15}$$

where $\dot{\mathbf{V}} \in \mathbb{R}^{6nm}$ is the link acceleration vector, $\ddot{\mathbf{q}} \in \mathbb{R}^{d_m}$ is the joint acceleration, $\dot{\mathbf{V}}_b \in \mathbb{R}^6$ is the base acceleration, and $\dot{\mathbf{X}}_b \in \mathbb{R}^{6nm\times 6}$ and $\dot{\mathbf{X}} \in \mathbb{R}^{6nm\times 6nm}$ are the time derivatives of $\mathbf{X}_b$ and $\mathbf{X}$, respectively. In this equation, $\dot{\mathbf{V}}_b$ is an unknown variable, whereas $\mathbf{V}_b$ is known.

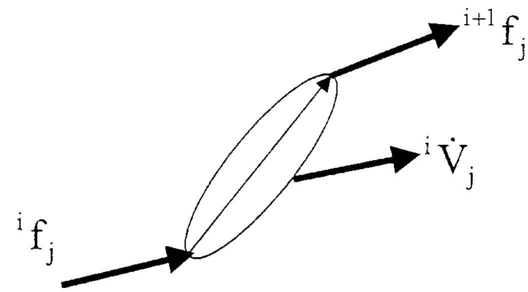

Figure 3. Free body diagram of the *i*th link of the *j*th robot.

From the FBD of the *i*th link of the *j*th robot as shown in Figure 3, the force exerted on the *i*th link of the *j*th robot in Cartesian space is expressed as

$${}^{i}\mathbf{f}_j - {}^{i}_{i+1}\mathbf{X}_j\,{}^{i+1}\mathbf{f}_j = {}^{i}\mathbf{M}_j\,{}^{i}\dot{\mathbf{V}}_j + {}^{i}\mathbf{b}_j \tag{16a}$$

$$= {}^{i}_{i+1}\mathbf{X}_j\,{}^{i+1}\mathbf{f}_j + {}^{i}\mathbf{M}_j\,{}^{i}\dot{\mathbf{V}}_j + {}^{i}\mathbf{b}_j, \tag{16b}$$

where ${}^{i}\mathbf{M}_j \subset \mathbb{R}^{6\times 6}$ is the spatial inertia, ${}^{i}\dot{\mathbf{V}}_j \in \mathbb{R}^6$ is the spatial acceleration, ${}^{i}\mathbf{f}_j \in \mathbb{R}^6$ is the spatial force, and ${}^{i}\mathbf{b}_j = (d/dt)({}^{i}\mathbf{M}_j){}^{i}\mathbf{V}_j$ is the bias force on the *i*th link of the *j*th robot. The bias force is the force applied to the rigid body to produce zero spatial acceleration. In Eq. (16), ${}^{i}_{i+1}\mathbf{X}_j$ projects the force acting on $(i+1)$th frame onto the *i*th link frame. In addition, any external forces that exist such as gravitational forces, etc. can be included at this stage in Eq. (16). However, to keep the mathematical relationships more clear and lucid, the effects of any external forces on the links are ignored.

In addition, Eq. (16) can be rearranged to be expressed in terms of the end-effector force ${}^{n+1}\mathbf{f}_j$ as

$${}^{i}\mathbf{f}_j = {}^{i}_{n+1}\mathbf{X}_j\,{}^{n+1}\mathbf{f}_j + \sum_{k=i}^{n} {}^{i}_{k}\mathbf{X}_j({}^{k}\mathbf{M}_j\,{}^{k}\dot{\mathbf{V}}_j + {}^{k}\mathbf{b}_j), \tag{17}$$

Here, the first term signifies the contribution of the end-effector force towards the force on a particular link frame and the second term inside the summation symbol represents the effects of all the link forces on the *i*th link starting from the last link.

Now the force vector for the jth robot can be described as

$$\mathbf{f}_j = \mathbf{X}_j^T(\mathbf{D}_j\, {}^{n+1}\mathbf{f}_j + \mathbf{M}_j\dot{\mathbf{V}}_j + \mathbf{b}_j), \tag{18}$$

where $\mathbf{f}_j = [{}^1\mathbf{f}_j^T\, {}^2\mathbf{f}_j^T \cdots {}^n\mathbf{f}_j^T]^T \in \mathbb{R}^{6n}$, $\dot{\mathbf{V}}_j = [{}^1\dot{\mathbf{V}}_j^T\, {}^2\dot{\mathbf{V}}_j^T \cdots {}^n\dot{\mathbf{V}}_j^T]^T \in \mathbb{R}^{6n}$, $\mathbf{b}_j = [{}^1\mathbf{b}_j^T\, {}^2\mathbf{b}_j^T \cdots {}^n\mathbf{b}_j^T]^T \in \mathbb{R}^{6n}$, $\mathbf{D}_j = [\mathbf{0} \cdots \mathbf{0}\, {}_n^{n+1}\mathbf{X}_j]^T = \mathbf{B}_j^T \in \mathbb{R}^{6n\times 6}$ and $\mathbf{M}_j = \mathrm{diag}({}^1\mathbf{M}_j\, {}^2\mathbf{M}_j \cdots {}^n\mathbf{M}_j) \in \mathbb{R}^{6n\times 6n}$.

Then, combining the terms in Eq. (18) for all the m robots yields

$$\mathbf{f} = \mathbf{X}^T(\mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b}) + \mathbf{X}^T\mathbf{D}\mathbf{f}_e, \tag{19}$$

where $\mathbf{f} = [\mathbf{f}_1^T\mathbf{f}_2^T \cdots \mathbf{f}_m^T]^T \in \mathbb{R}^{6nm}$, $\dot{\mathbf{V}} = [\dot{\mathbf{V}}_1^T\dot{\mathbf{V}}_2^T \cdots \dot{\mathbf{V}}_m^T]^T \in \mathbb{R}^{6nm}$, $\mathbf{b} = (d/dt)(\mathbf{M}_q)\mathbf{V} = [\mathbf{b}_1^T\mathbf{b}_2^T \cdots \mathbf{b}_m^T]^T \in \mathbb{R}^{6nm}$, $\mathbf{D} = \mathrm{diag}(\mathbf{D}_1\mathbf{D}_2 \cdots \mathbf{D}_m) \in \mathbb{R}^{6nm\times 6nm}$, $\mathbf{M}_q = \mathrm{diag}(\mathbf{M}_1\mathbf{M}_2 \cdots \mathbf{M}_m) \in \mathbb{R}^{6nm\times 6nm}$, and $\mathbf{f}_e = [{}^{n+1}\mathbf{f}_1^T\, {}^{n+1}\mathbf{f}_2^T \cdots {}^{n+1}\mathbf{f}_m^T]^T \in \mathbb{R}^{6m}$ is the force exerted by the end-effector on the object or any external force acting on the end-effector. Here, $\mathbf{X}^T$ is upper block triangular as opposed to $\mathbf{X}$ which is lower block triangular. Hence, $\mathbf{X}^T$ is *anticausal*.[18] $\mathbf{X}^T$ propagates link forces from the end-effector to the base that can be stated as the anticausal direction in contrast to the base-to-end-effector propagation of $\mathbf{X}$, which is known as *causal*. In addition, $\mathbf{M}_q$ is the block diagonal and so can be interpreted as a memoryless operator.

Now, Eq. (19) can be rewritten as

$$\mathbf{f} - \mathbf{X}^T\mathbf{D}\mathbf{f}_e = \mathbf{X}^T(\mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b}). \tag{20}$$

Here, the action of the expression $\mathbf{X}^T\mathbf{D}$ on $\mathbf{f}_e$ is as follows: (i) $\mathbf{D}$ drags the end-effector forces of each manipulator $\mathbf{f}_j^e$ to its last link frame, which can be represented as $[\mathbf{0}\,\mathbf{0} \cdots {}^n\mathbf{f}_1^T\,\mathbf{0}\,\mathbf{0} \cdots {}^n\mathbf{f}_2^T \cdots \mathbf{0}\,\mathbf{0} \cdots {}^n\mathbf{f}_m^T]^T \in \mathbb{R}^{6nm}$; ($ii$) then $\mathbf{X}^T$ propagates each of these forces anticausally from nth link frame to the first link frame forming the interaction spatial forces between neighboring links, and is represented as $\mathbf{f}' = [{}^1\mathbf{f}_1^T\, {}^2\mathbf{f}_1^T \cdots {}^n\mathbf{f}_1^T\, {}^1\mathbf{f}_2^T\, {}^2\mathbf{f}_2^T \cdots {}^n\mathbf{f}_2^T \cdots {}^1\mathbf{f}_m^T\, {}^2\mathbf{f}_m^T \cdots {}^n\mathbf{f}_m^T]^T \in \mathbb{R}^{6nm}$. Then, the remaining expression $(\mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b})$ defines the link forces due to their motion. The action of $\mathbf{X}^T$ on $(\mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b})$ is to represent the interaction spatial forces by propagating all the single link forces due to their motion anticausally to the respective link frames.

From Eq. (17), the force exerted by the base on the first link of the jth robot can be expressed as

$${}^1\mathbf{f}_j = {}_{n+1}^{\;\;1}\mathbf{X}_j\, {}^{n+1}\mathbf{f}_j + \sum_{k=1}^{n} {}_k^1\mathbf{X}_j({}^k\mathbf{M}_j\, {}^k\dot{\mathbf{V}}_j + {}^k\mathbf{b}_j).$$

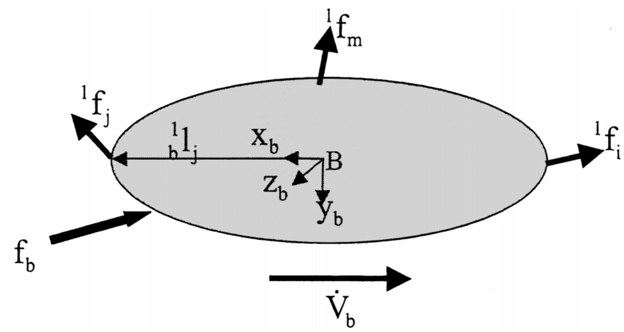

Figure 4. Free body diagram of the base of the multi-arm robotic system.

This force vector ${}^1\mathbf{f}_j$ is nothing but the first equation in the equation set derived in Eq. (18), which can alternatively be represented as

$${}^1\mathbf{f}_j = [{}_1^1\mathbf{X}_j\, {}_2^1\mathbf{X}_j \cdots {}_n^1\mathbf{X}_j](\mathbf{D}\mathbf{f}_e + \mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b}). \tag{21}$$

5.2. Base Dynamics

The objective of this subsection is to find expressions for both $\dot{\mathbf{V}}$ and $\dot{\mathbf{V}}_b$ in terms of $\mathbf{f}_b$, $\mathbf{f}_e$ and $\ddot{\mathbf{q}}$. Now, the force equilibrium equation of the base from its FBD, as illustrated in Figure 4, can be represented as

$$\mathbf{f}_b - \sum_{j=1}^{m} {}_1^b\mathbf{X}_j\, {}^1\mathbf{f}_j = \mathbf{M}_b\dot{\mathbf{V}}_b + \mathbf{b}_b, \tag{22}$$

where $\mathbf{f}_b \in \mathbb{R}^6$ is the external base force acting on the CM of the base, $\mathbf{M}_b \in \mathbb{R}^{6\times 6}$ is the base inertial matrix, $\dot{\mathbf{V}}_b \in \mathbb{R}^6$ is the base acceleration, and $\mathbf{b}_b = (d/dt) \times (\mathbf{M}_b)\mathbf{V}_b \in \mathbb{R}^6$ is the base bias force. By substituting the value of ${}^1\mathbf{f}_j$ from Eq. (21) into Eq. (22) and rearranging the terms, the modified expression for $\mathbf{f}_b$ is given as

$$\begin{aligned}\mathbf{f}_b &= \mathbf{M}_b\dot{\mathbf{V}}_b + \mathbf{b}_b + \sum_{j=1}^{m} {}^b\mathbf{X}_j^T(\mathbf{D}\mathbf{f}_e + \mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b}) \\ &= \mathbf{M}_b\dot{\mathbf{V}}_b + \mathbf{b}_b + \mathbf{X}_b^T(\mathbf{D}\mathbf{f}_e + \mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b}).\end{aligned} \tag{23}$$

Now, solving Eq. (23) for the base acceleration gives

$$\dot{\mathbf{V}}_b = \mathbf{M}_b^{-1}\{\mathbf{f}_b - \mathbf{b}_b - \mathbf{X}_b^T(\mathbf{D}\mathbf{f}_e + \mathbf{M}_q\dot{\mathbf{V}} + \mathbf{b})\}. \tag{24}$$

Then, substituting the value of $\dot{\mathbf{V}}_b$ in Eq. (15), the acceleration of the manipulators can be expressed as

$$\dot{\mathbf{V}}=(\mathbf{E}_{6nm}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{M}_q)^{-1}\{\mathbf{X}\boldsymbol{\Phi}\ddot{\mathbf{q}}+\dot{\mathbf{X}}_b\mathbf{V}_b - \mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T(\mathbf{D}\mathbf{f}_e+\mathbf{b})+\mathbf{X}_b\mathbf{M}_b^{-1}(\mathbf{f}_b-\mathbf{b}_b)+\dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}}\}. \tag{25}$$

Hence, with known external force and base velocity, $\dot{\mathbf{V}}$ can be computed for each link of all the manipulators.

Now, substituting the value of $\dot{\mathbf{V}}$ in Eq. (25) into Eq. (24), we have

$$\dot{\mathbf{V}}_b=\mathbf{M}_b^{-1}\{(\mathbf{E}_6-\mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1})(\mathbf{f}_b-\mathbf{b}_b)-\mathbf{X}_b^T(\mathbf{E}_{6nm} - \mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)\mathbf{D}\mathbf{f}_e-\mathbf{X}_b^T\mathbf{G}\mathbf{X}\boldsymbol{\Phi}\ddot{\mathbf{q}}-\mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}_b\mathbf{V}_b + \mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{b}-\mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}}-\mathbf{X}_b^T\mathbf{b}\}, \tag{26}$$

where $\mathbf{G}=(\mathbf{M}_q^{-1}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)^{-1}=\mathbf{M}_q(\mathbf{E}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{M}_q)^{-1}$.

The inversion of the matrix can be simplified using the matrix inversion lemma, which gives

$$(\mathbf{M}_q^{-1}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)^{-1}=\mathbf{M}_q-\mathbf{M}_q\mathbf{X}_b(\mathbf{M}_b + \mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b)^{-1}\mathbf{X}_b^T\mathbf{M}_q .$$

Now, in this simplified form, the maximum order of the matrix to be inverted has decreased to a 6×6 matrix.

5.3. Equations of Motion

Here, a mathematical expression for the joint torque is presented in terms of $\mathbf{f}_e$ and $\ddot{\mathbf{q}}$. Now, an explicit relationship between the force and joint position, velocity, and acceleration can be obtained by eliminating $\dot{\mathbf{V}}$ from Eqs. (25) and (20):

$$\mathbf{f}-\mathbf{X}^T(\mathbf{M}_q^{-1}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)^{-1}\mathbf{M}_q^{-1}\mathbf{D}\mathbf{f}_e =\mathbf{X}^T(\mathbf{M}_q^{-1}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)^{-1}\{\mathbf{X}\boldsymbol{\Phi}\ddot{\mathbf{q}}+\dot{\mathbf{X}}_b\mathbf{V}_b+\mathbf{M}_q^{-1}\mathbf{b} + \mathbf{X}_b\mathbf{M}_b^{-1}(\mathbf{f}_b-\mathbf{b}_b)+\dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}}\} \tag{27}$$

The active joint torque $\mathbf{T}$ can be obtained by multiplying both sides of Eq. (27) with $\boldsymbol{\Phi}^T$, which gives

$$\boldsymbol{\Phi}^T\mathbf{f}-\boldsymbol{\Phi}^T\mathbf{X}^T\{\mathbf{M}_q-\mathbf{M}_q\mathbf{X}_b(\mathbf{M}_b + \mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b)^{-1}\mathbf{X}_b^T\mathbf{M}_q\}\mathbf{M}_q^{-1}\mathbf{D}\mathbf{f}_e =\boldsymbol{\Phi}^T\mathbf{X}^T\{\mathbf{M}_q-\mathbf{M}_q\mathbf{X}_b(\mathbf{M}_b+\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b)^{-1}\mathbf{X}_b^T\mathbf{M}_q\} \times\{\mathbf{X}\boldsymbol{\Phi}\ddot{\mathbf{q}}+\dot{\mathbf{X}}_b\mathbf{V}_b+\mathbf{M}_q^{-1}\mathbf{b} + \mathbf{X}_b\mathbf{M}_b^{-1}(\mathbf{f}_b-\mathbf{b}_b)+\dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}}\}.$$

This yields the final expression for the joint torque vector, which can be concisely represented as

$$\hat{\mathbf{T}}-\mathbf{J}^T\mathbf{f}_e=\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}, \tag{28}$$

where the generalized inertia tensor $\mathbf{M}=\mathbf{M}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{M}_{ap}$ with $\mathbf{M}_a=\boldsymbol{\Phi}^T\mathbf{X}^T\mathbf{M}_q\mathbf{X}\boldsymbol{\Phi}$, $\mathbf{M}_{ap}^T=\boldsymbol{\Phi}^T\mathbf{X}^T\mathbf{M}_q\mathbf{X}_b$, $\mathbf{M}_p=\mathbf{M}_b+\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b$ and $\mathbf{M}_{ap}=\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}\boldsymbol{\Phi}$; $\mathbf{J}$ is the generalized Jacobian matrix and its transpose is $\mathbf{J}^T=\mathbf{J}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{J}_p$ with $\mathbf{J}_a=\boldsymbol{\Phi}^T\mathbf{X}^T\mathbf{D}$ and $\mathbf{J}_p=\mathbf{X}_b^T\mathbf{D}$; the coriolis and centrifugal force vector $\mathbf{C}=\mathbf{C}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{C}_p$ with $\mathbf{C}_a=\boldsymbol{\Phi}^T\mathbf{X}^T\mathbf{b}+\boldsymbol{\Phi}^T\mathbf{X}^T\mathbf{M}_q\dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}}$, $\mathbf{C}_p=\mathbf{X}_b^T(\mathbf{b}+\mathbf{M}_q\dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}})-(\mathbf{M}_b\mathbf{X}_b^{-1}\dot{\mathbf{X}}_b\mathbf{V}_b-\mathbf{b}_b)$, and $\hat{\mathbf{T}}=\mathbf{T}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{T}_p$ with $\mathbf{T}_p=\mathbf{f}_b$, and $\mathbf{T}=\mathbf{T}_a=\boldsymbol{\Phi}^T\mathbf{f}$. The representation in Eq. (28) is also known as the inverse dynamics equation or the torque equation. This equation yields the values of $\mathbf{f}_b$ and $\mathbf{T}_a$ from a given $\ddot{\mathbf{q}}$ and $\mathbf{f}_e$.

In the above representation the subscript "a" and "p" stands for active and passive elements, respectively. The notion of active and passive has been introduced to make the dynamic analysis more general. In a space robot, the base serves as a passive joints, which in its generalized representation includes all passive joint in the system. A thorough physical interpretation of Eq. (28) has been presented in Section 6. The derivation of these coefficient matrices and vectors are shown in Appendix B. In addition, $\mathbf{M}$ is a symmetric positive definite matrix and is also known as the system mass matrix.

5.4. Object Dynamics

This subsection deals with the computation of $\mathbf{f}_e$ from the knowledge of the object acceleration $\dot{\mathbf{V}}^o$ and its mechanical parameters. An object is assumed to be held rigidly by m manipulators. The FBD of the object is shown in Figure 5. Then, the net generalized force at the center of mass of the object, due to all the end-effector forces acting on it, can be represented as[23,24]

$$\mathbf{f}_o=\mathbf{W}^T\mathbf{f}_e, \tag{29}$$

where $\mathbf{W}^T=[{}^{n+1}_{n+2}\mathbf{X}_1^T\,{}^{n+1}_{n+2}\mathbf{X}_2^T\cdots{}^{n+1}_{n+2}\mathbf{X}_m^T]\in\mathbb{R}^{6\times 6m}$, with $(n+2)$th and $(n+1)$th joints representing object center of mass and end-effector contact point with the object, respectively; $\mathbf{f}_o=[\eta_o^T\, f_o^T]^T\in\mathbb{R}^6$, with $f_o\in\mathbb{R}^3$ and $\eta_o\in\mathbb{R}^3$ the force and moment vectors at the object center of mass. The matrix $\mathbf{W}$ is known as the grip matrix or grasp matrix. This is a positive definite ma-

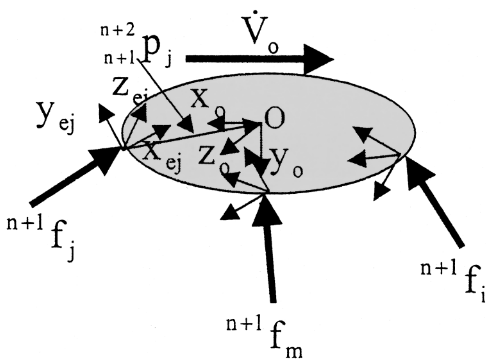

Figure 5. Free body diagram of the object held by m manipulators.

trix and thus nonsingular. Now, the transformation matrix ${}^{n+1}_{n+2}\mathbf{X}^T_j = {}^{n+2}_{n+1}\mathbf{X}_j$ transforms the contact force of the jth manipulator to its equivalent at the center of mass of the object. This can be defined as

$$ {}^{n+2}_{n+1}\mathbf{X}_j = \begin{bmatrix} \mathbf{E}_3 & \mathbf{0}_3 \\ \zeta_j(\mathbf{q}_j) & \mathbf{E}_3 \end{bmatrix}, \tag{30}$$

where $\mathbf{E}_3$ and $\mathbf{0}_3$ are 3×3 identity and zero matrices, respectively. The 3×3 matrix $\zeta_j(\mathbf{q}_j)$ arises from the cross-product operator ${}^{n+2}_{n+1}p_j\times$, where ${}^{n+2}_{n+1}p_j = [{}^{n+2}_{n+1}p_{j,x}\, {}^{n+2}_{n+1}p_{j,y}\, {}^{n+2}_{n+1}p_{j,z}]^T$ is the moment arm from the jth end-effector contact point to the center of mass of the object represented in the base frame. This can be defined as

$$\zeta_j(\mathbf{q}_j) = \begin{bmatrix} 0 & {}^{n+2}_{n+1}p_{j,z} & -{}^{n+2}_{n+1}p_{j,y} \\ -{}^{n+2}_{n+1}p_{j,z} & 0 & {}^{n+2}_{n+1}p_{j,x} \\ {}^{n+2}_{n+1}p_{j,y} & -{}^{n+2}_{n+1}p_{j,x} & 0 \end{bmatrix}. \tag{31}$$

The force balance equation for this object from its FBD in Figure 5, can be represented as

$$\mathbf{f}_o = \mathbf{M}_o \dot{\mathbf{V}}^o + \mathbf{b}_o\,, \tag{32}$$

where $\mathbf{M}_o \in \mathbb{R}^{6\times 6}$ is the object inertia matrix and $\mathbf{b}_o = (d/dt)(\mathbf{M}_o)\mathbf{V}^o \in \mathbb{R}^6$ is the bias force on the object required to produce zero object acceleration.

Now combining Eqs. (29) and (32), the following dynamic equation for the object can be obtained:

$$\mathbf{M}_o \dot{\mathbf{V}}^o + \mathbf{b}_o = \mathbf{W}^T \mathbf{f}_e\,. \tag{33}$$

6. PHYSICAL INTERPRETATION OF THE INVERSE DYNAMICS EQUATION

The inverse dynamics equation as represented in Eq. (28) consists of a mass matrix term $\mathbf{M}$ due to all active joints $\mathbf{M}_a$ and an expression consisting of dynamically coupled active and passive joint variables. The mass matrix factorization of operator $\mathbf{M}_a$ due to active joints is also called the Newton–Euler factorization,[18] as it establishes the equivalence between Lagrangian and Newton–Euler formulations of manipulator dynamics. The action of $\mathbf{M}_a$ can be described as follows: (i) $\mathbf{\Phi}$ acts on the joint acceleration $\ddot{\mathbf{q}}$ to result in a vector of relative spatial accelerations between the links; (ii) then $\mathbf{X}$ acts on $\mathbf{\Phi}\ddot{\mathbf{q}}$ in a causal way to propagate link relative accelerations to obtain spatial accelerations of all the links; (iii) then the memoryless operator $\mathbf{M}_q$ acts on $\mathbf{X\Phi}\ddot{\mathbf{q}}$ to represent all spatial forces on each of the links; (iv) then $\mathbf{X}^T$ acts on $\mathbf{M}_q\mathbf{X\Phi}\ddot{\mathbf{q}}$ for the propagation of all the spatial forces to form the link interaction spatial forces; and (v) finally, the action of the memoryless operator $\mathbf{\Phi}^T$ on link interaction spatial forces is to project each of these forces to the joint axes, thereby resulting in joint active torques $\mathbf{T}_a$.

Now the action of the second expression $\mathbf{M}^T_{ap}\mathbf{M}_p^{-1}\mathbf{M}_{ap}$ of the system mass matrix on the joint acceleration can be interpreted as follows: (i) the expression $\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b$ inside $\mathbf{M}_p^{-1}$ represents the transformation of the quantity $\mathbf{M}_q$ to the base co-ordinate frames. Thus, $\mathbf{M}_p$ signifies the effective mass of the entire system projected onto the base co-ordinate system; (ii) analogous to the reasoning in active variables, $\mathbf{M}_{ap}\ddot{\mathbf{q}}$ projects all spatial forces on each of the links onto the base interaction spatial force; (iii) then $\mathbf{M}_p^{-1}$ acts on $\mathbf{M}_{ap}\ddot{\mathbf{q}}$ to represent the base spatial acceleration corresponding to the base spatial force; (iv) then $\mathbf{X}_b$ operates on $\mathbf{M}_p^{-1}\mathbf{M}_{ap}\ddot{\mathbf{q}}$ to propagate the base spatial acceleration into all the active links; (v) then the memoryless operator $\mathbf{M}_q$ acts on $\mathbf{X}_b\mathbf{M}_p^{-1}\mathbf{M}_{ap}\ddot{\mathbf{q}}$ to represent all spatial forces on each of the active links; (vi) then $\mathbf{X}^T$ operates to propagate anticausally all the single link spatial forces to form link interaction spatial forces; and (vii) finally, the memoryless operator $\mathbf{\Phi}^T$ acts on the link interaction forces to project each of them onto their respective joint axes to obtain the joint torques.

Thus, the action of $\mathbf{M}^T_{ap}\mathbf{M}_p^{-1}\mathbf{M}_{ap}$ can be precisely summarized as follows. The active joint accelerations $\ddot{\mathbf{q}}$ are converted to the effective forces projected onto the passive co-ordinate by the operator $\mathbf{M}_{ap}$, and subsequently converted into equivalent accelerations in passive co-ordinates by the action of $\mathbf{M}_p^{-1}$. Then, these accelerations are projected back onto the active components through $\mathbf{M}_{ap}^T$ in the shape of active torques on the joints. This shows that by this process the dynamic interaction between active and passive elements are introduced into the system equation.

The system Jacobian matrix $\mathbf{J}^T$ also consists of one active variable component, and one dynamic coupling component representing the interaction among active and passive variables. The factorization of the active components of the generalized Jacobian matrix $\mathbf{J}_a = \mathbf{\Phi}^T \mathbf{X}^T \mathbf{D} = \mathbf{J}_q^T$ clearly satisfies the well known dual relationship between velocity and force.[25] The action of $\mathbf{J}_a$ on $\mathbf{f}_e$ can be summarized as follows: (*i*) the action of $\mathbf{D}$ on $\mathbf{f}_e$ maps it into the column vector as $[\mathbf{0}\,\mathbf{0}\cdots{}^n\mathbf{f}_1^T\mathbf{0}\,\mathbf{0}\cdots{}^n\mathbf{f}_2^T\cdots\mathbf{0}\,\mathbf{0}\cdots{}^n\mathbf{f}_m^T]^T$; (*ii*) then $\mathbf{X}^T$ acts on each of these forces at the last link frame of each manipulator to propagate those from the *n*th link frame to the first link frame (thus constituting the interaction spatial forces between neighboring links); and (*iii*) finally, $\mathbf{\Phi}^T$ projects each component of the interaction spatial force onto the respective joint axes to obtain the joint torques. This effectively shows how the end-effector force affects each of the active joints and vice versa.

Then, the action of the second component of $\mathbf{J}^T$, which is represented by $\mathbf{M}_{ap}^T \mathbf{M}_p^{-1} \mathbf{J}_p$, can be explained as follows: (*i*) first, to analyze the action of $\mathbf{J}_p$ on $\mathbf{f}_e$, it can be observed that $\mathbf{D}$ takes the end-effector forces $\mathbf{f}_e$ onto the last link frame of each manipulator and then $\mathbf{X}_b^T$ acts on it to project all these forces onto the passive frame to represent the effective spatial force vector in that frame; (*ii*) then $\mathbf{M}_p^{-1}$ maps the effective spatial force at the passive co-ordinates due to $\mathbf{f}_e$ into effective spatial acceleration in those co-ordinates; and (*iii*) finally, $\mathbf{M}_{ap}^T$, through successive transformations by its individual component factors as described in the preceding explanation of mass matrix, projects the effective passive spatial accelerations into joint torques of the active joints.

The coriolis and centrifugal force is also grouped into purely active and coupled terms, which can be interpreted as follows: (*i*) the purely active component can be analyzed with the action of $\mathbf{X}^T$ that acts on the net active link bias force vector $\mathbf{b}$ to represent the active link interaction bias force, and then $\mathbf{\Phi}^T$ acts on that to project $\mathbf{X}^T\mathbf{b}$ onto joint axes to contribute towards joint torques; (*ii*) the dynamic coupling term $\mathbf{C}_p$ represents the effective bias force due to the mutual interaction between the links and the base. The role of $\mathbf{C}_p$ can be analyzed by considering separately both of its constituent expressions $\mathbf{X}_b^T(\mathbf{b} + \mathbf{M}_q \dot{\mathbf{X}} \mathbf{\Phi} \dot{\mathbf{q}})$ and $(\mathbf{M}_b \mathbf{X}_b^{-1} \dot{\mathbf{X}}_b \mathbf{V}_b - \mathbf{b}_b)$. Here, the former is the projection of the bias forces of the active joints onto the passive joints, and the latter is purely a passive bias force. Now, the analysis of the active part can be progressed as follows: (*i*) the diagonal memoryless operator $\mathbf{\Phi}$ acts on $\dot{\mathbf{q}}$ to result in a vector of relative spatial velocities between the manipulator links; (*ii*) then the action $\dot{\mathbf{X}}$ is to transform the spatial velocities $\mathbf{\Phi}\dot{\mathbf{q}}$ into relative spatial accelerations between the links and subsequently to propagate these relative accelerations causally from base to tip to result in biases in the link spatial accelerations; (*iii*) $\mathbf{M}_q$ then acts on the link bias accelerations to represent spatial bias forces on each of the links; (*iv*) the remaining direct spatial bias force on each of the links due to their spatial momentum is added to the previously calculated spatial bias forces on the active joints; (*v*) then, $\mathbf{X}_b^T$ projects the net spatial bias forces in active joint space to passive joint space. Then, the remaining passive parts can be analyzed as follows: (*i*) $\dot{\mathbf{X}}_b$ acts on $\mathbf{V}_b$ to represent the spatial bias acceleration on the passive joints due to their motion, and propagates these into the active joint space to represent the bias accelerations; (*ii*) then $\mathbf{X}_b^{-1}$ acts on $\dot{\mathbf{X}}_b \mathbf{V}_b$ to project the bias accelerations in the active space to passive space; (*iii*) the action of $\mathbf{M}_b$ then represents the spatial bias forces on the passive joints; (*iv*) then the difference of this bias force and that due to passive body momentum results in the net bias forces due to passive bodies alone.

Now, it can be observed that the difference between the active joint bias forces and passive joint bias force on the passive platform yields the effective passive bias force as represented in the passive coordinates. As discussed earlier, the action of $\mathbf{M}_p^{-1}$ on $\mathbf{C}_p$ is to convert the effective base interaction link bias force into spatial accelerations. Then, $\mathbf{M}_{ap}^T$ projects back these accelerations onto the active space in the shape of active joint torques, thereby representing the effect of interaction among links and base on final control torque.

Hence, the overall role of coriolis and centrifugal force vectors can be interpreted as the net effect of active bias forces, passive bias forces and interaction bias forces on the active torque to control the system.

The last component to be analyzed is the system torque $\hat{\mathbf{T}}$. Usually, this consists of a purely active part, and an interaction part. The purely active torque $\mathbf{T}_a = \mathbf{\Phi}^T \mathbf{f}$ represents the joint torques due to end-effector force and motion of the individual links without any passive variables in the system. However, when passive components are present that are subject to known desired external forces, they alter the overall torque expression. Hence, the effects of an external force $\mathbf{f}_b$ on the passive joints (i.e., moving base in a space manipulator system) can be analyzed as follows: (*i*) $\mathbf{M}_p^{-1}$ acts on $\mathbf{T}_p = \mathbf{f}_b$ to transform these forces into the spatial interaction acceleration in the passive body frames; (*ii*) then $\mathbf{M}_{ap}^T$ projects these interaction accelerations in the passive frame onto the active frames by transforming them into joint torques.

From the above analysis, it has been observed that one common expression dictates the role of passive components and their interaction with the active components. This expression is $\mathbf{M}_{ap}^T \mathbf{M}_p^{-1}$, known as the *dynamic coupling* expression. This dynamic coupling expression maps the mutual interaction terms between active and passive variables (mass matrix, coriolis and centrifugal force, and system torques) onto the active variables. Hence, the role of the dynamic coupling factor can be expressed by two distinct factors. The factor $\mathbf{M}_p = \mathbf{M}_b + \mathbf{X}_b^T \mathbf{M}_q \mathbf{X}_b$ represents the effective spatial inertia of the complete system represented in the passive frame. Hence, $\mathbf{M}_p^{-1}$ is always used to transform a spatial force into the effective spatial acceleration of the passive bodies. Then, as analyzed before, the action of $\mathbf{M}_{ap}^T$ on these accelerations is to transform them into their respective contributions towards the final torques by projecting all onto active space.

7. FORWARD DYNAMICS

The forward dynamics analysis of a co-operating manipulator system holding a common object mounted on a mobile platform can be described with reference to Eq. (28), which involves the computation of the joint accelerations $\ddot{\mathbf{q}}$ with the knowledge of the input torques and forces, $\mathbf{T}$ and $\mathbf{f}_b$, current state of the manipulator, $\mathbf{q}$, $\dot{\mathbf{q}}$, and motion of the base.

The end-effector velocity can be expressed in terms of the object velocity with the use of principle of virtual work that establishes the duality between forces and velocities:

$$\mathbf{V}^e = \mathbf{W}\mathbf{V}^o, \tag{34}$$

where $\mathbf{V}^o \in \mathbb{R}^6$ is the velocity of the center of mass of the object.

The end-effector acceleration $\dot{\mathbf{V}}^e$ can be obtained by differentiating Eq. (34) with respect to time:

$$\dot{\mathbf{V}}^e = \mathbf{W}\dot{\mathbf{V}}^o + \dot{\mathbf{W}}\mathbf{V}^o. \tag{35}$$

Here, $\dot{\mathbf{V}}^o \in \mathbb{R}^6$ is the object spatial acceleration.

Another representation for $\dot{\mathbf{V}}^e$ can be obtained from Eq. (14):

$$\dot{\mathbf{V}}^{\mathbf{o}} = \mathbf{J}_b \dot{\mathbf{V}}_b + \mathbf{J}_q \ddot{\mathbf{q}} + \dot{\mathbf{J}}_b \mathbf{V}_b + \dot{\mathbf{J}}_q \dot{\mathbf{q}}. \tag{36}$$

Now, the base acceleration $\dot{\mathbf{V}}_b$ in Eq. (26) can be represented as

$$\dot{\mathbf{V}}_b = \mathbf{J}_D \ddot{\mathbf{q}} + \text{terms not containing acceleration}, \tag{37}$$

where

$$\mathbf{J}_D = \mathbf{M}_p^{-1} \mathbf{M}_{ap}. \tag{38}$$

Here $\mathbf{J}_D$ is also called as the *disturbance Jacobian* (derivation is given in Appendix B). In the expression of $\mathbf{J}_D$, the factor $\mathbf{M}_p^{-1}\mathbf{M}_{ap}$, which is the transpose of the dynamic coupling factor, acts on the active joint motion $\ddot{\mathbf{q}}$ to transform it into spatial accelerations of the passive body represented in the passive frame. Hence, the disturbance Jacobian describes the incremental motion of passive joints. This incremental motion of passive joints is solely due to the disturbance on the passive joints by the active joints.

Substituting the values for $\dot{\mathbf{V}}_b$ as in Eq. (26) into Eq. (36), we have

$$\begin{aligned}\dot{\mathbf{V}}^e = {} & \mathbf{J}_b \mathbf{M}_b^{-1}\{(\mathbf{E}_6 - \mathbf{X}_b^T \mathbf{G} \mathbf{X}_b \mathbf{M}_b^{-1})(\mathbf{f}_b - \mathbf{b}_b) - \mathbf{X}_b^T \mathbf{G} \dot{\mathbf{X}}_b \mathbf{V}_b \\ & + \mathbf{X}_b^T \mathbf{G} \mathbf{X}_b \mathbf{M}_b^{-1} \mathbf{X}_b^T \mathbf{b} - \mathbf{X}_b^T \mathbf{b}\} + (\mathbf{J}_q - \mathbf{J}_b \mathbf{J}_D)\ddot{\mathbf{q}} + \dot{\mathbf{J}}_b \mathbf{V}_b \\ & + (\dot{\mathbf{J}}_q - \mathbf{J}_b \mathbf{M}_b^{-1} \mathbf{X}_b^T \mathbf{G} \dot{\mathbf{X}} \mathbf{\Phi})\dot{\mathbf{q}} \\ & - \mathbf{J}_b \mathbf{M}_b^{-1} \mathbf{X}_b^T (\mathbf{E}_{6nm} - \mathbf{G} \mathbf{X}_b \mathbf{M}_b^{-1} \mathbf{X}_b^T) \mathbf{D} \mathbf{f}_e.\end{aligned} \tag{39}$$

Now, Eq. (39) can be represented as

$$\dot{\mathbf{V}}^e = \mathbf{J}\ddot{\mathbf{q}} + \text{terms not containing acceleration}, \tag{40}$$

where

$$\mathbf{J} = \mathbf{J}_q - \mathbf{J}_b \mathbf{J}_D. \tag{41}$$

Here, $\mathbf{J}$ is called the generalized Jacobian matrix. Hence, $\mathbf{J}$ is represented by the difference between the pure active link Jacobian and the disturbance Jacobian projected onto the active space by the base Jacobian. Expressed another way, the disturbance caused by the movements of the active joints on the passive joints is reflected back onto the active space to change its motion. Thus, it shows the strong mutual dependency of passive and active variables.

Now, $\dot{\mathbf{V}}^e$ in Eq. (39) can be represented as

$$\begin{aligned}\dot{\mathbf{V}}^e = &\mathbf{J}_b\mathbf{M}_b^{-1}\{(\mathbf{E}_6 - \mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1})(\mathbf{f}_b - \mathbf{b}_b) - \mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}_b\mathbf{V}_b \\ &+ \mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{b}\mathbf{X}_b^T\mathbf{b}\} + \mathbf{J}\mathbf{M}^{-1}(\mathbf{T} - \mathbf{C}) + \dot{\mathbf{J}}_b\mathbf{V}_b \\ &+ (\dot{\mathbf{J}}_q - \mathbf{J}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}\mathbf{\Phi})\dot{\mathbf{q}} - \{(\mathbf{J}_q - \mathbf{J}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{G}\mathbf{X}\mathbf{\Phi}) \\ &\times \mathbf{M}^{-1}\mathbf{J}^T + \mathbf{J}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T(\mathbf{E}_{6nm} - \mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)\mathbf{D}\}\mathbf{f}_e. \end{aligned} \tag{42}$$

However, $\dot{\mathbf{V}}^e$ can also be expressed as

$$\dot{\mathbf{V}}^e = \dot{\mathbf{V}}^e_{\text{open}} - \dot{\mathbf{V}}^e_{\text{constrained}}, \tag{43}$$

where

$$\begin{aligned}\dot{\mathbf{V}}^e_{\text{open}} = &\mathbf{J}_b\mathbf{M}_b^{-1}\{(\mathbf{E}_6 - \mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1})(\mathbf{f}_b - \mathbf{b}_b) - \mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}_b\mathbf{V}_b \\ &+ \mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{b}\mathbf{X}_b^T\mathbf{b}\} + \mathbf{J}\mathbf{M}^{-1}(\mathbf{T} - \mathbf{C}) + \dot{\mathbf{J}}_b\mathbf{V}_b \\ &+ (\dot{\mathbf{J}}_q - \mathbf{J}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}\mathbf{\Phi})\dot{\mathbf{q}} \end{aligned} \tag{44}$$

and

$$\begin{aligned}\dot{\mathbf{V}}^e_{\text{constrained}} = &\{\mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T + \mathbf{J}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T \\ &\times (\mathbf{E}_{6nm} - \mathbf{G}\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{E}_{6nm})\mathbf{D}\}\mathbf{f}_e. \end{aligned} \tag{45}$$

Hence, the system can be modeled as a superposition of an open chain part and a constrained part due to the presence of co-operation.

Now, using Eqs. (35) and (43), it is possible to find an explicit relationship between the end-effector force and object acceleration, which gives

$$\dot{\mathbf{V}}^e_{\text{constrained}} = \dot{\mathbf{V}}^e_{\text{open}} - \mathbf{W}\dot{\mathbf{V}}^o - \dot{\mathbf{W}}\mathbf{V}^o. \tag{46}$$

Then, the end-effector force vector can be expressed as

$$\mathbf{f}_e = \mathbf{H}^{-1}(\dot{\mathbf{V}}^e_{\text{open}} - \mathbf{W}\dot{\mathbf{V}}^o - \dot{\mathbf{W}}\mathbf{V}^o), \tag{47}$$

where

$$\mathbf{H} = \mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T + \mathbf{J}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T(\mathbf{E}_{6nm} - \mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)\mathbf{D}. \tag{48}$$

In Eq. (48), the matrix $\mathbf{H} \in \mathbb{R}^{6m\times 6m}$ is a square, nonsingular matrix and it plays an important role in the robot dynamics calculations, as this yields unique end-effector forces with given values of $\mathbf{f}_b$ and $\mathbf{T}_a$. Now, substituting the value of $\mathbf{f}_e$ from Eq. (47) into Eq. (33) gives

$$\dot{\mathbf{V}}^o = (\mathbf{M}_o^{-1} + \mathbf{W}^T\mathbf{H}^{-1}\mathbf{W})^{-1}\{\mathbf{H}^{-1}(\dot{\mathbf{V}}^e_{\text{open}} - \dot{\mathbf{W}}\mathbf{V}^o) - \mathbf{b}_o\}. \tag{49}$$

Once the spatial acceleration of the object $\dot{\mathbf{V}}^o$ is calculated from Eq. (49), then Eq. (47) can give all the end-effector spatial forces.

8. CASE STUDIES

Now in this section, the equations of motion for free-base space robots and fixed-base robots are presented as a special case of the generalized formulation. Further, the generality of the above equations of motion for other cases such as flexible base and flexible arm manipulators is discussed in Appendix C.

8.1. Fixed-Base Manipulator

For a fixed base, fully actuated, rigid link manipulator system, the variables in Eq. (28) can be set to the following values

$$\mathbf{M}_o \to \infty, \quad \mathbf{V}_b \to \infty, \quad \text{and} \quad \mathbf{f}_b = \mathbf{0}.$$

Hence, now the dynamics of a fixed base manipulator system becomes

$$\mathbf{T}_a - \mathbf{J}^T\mathbf{f}_e = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C},$$

$$\dot{\mathbf{V}}^e = \mathbf{W}\dot{\mathbf{V}}^o + \dot{\mathbf{W}}\mathbf{V}^o = \mathbf{J}_q\ddot{\mathbf{q}} + \dot{\mathbf{J}}_q\dot{\mathbf{q}},$$

$$\mathbf{M}_o\dot{\mathbf{V}}^o + \mathbf{b}_o = \mathbf{W}^T\mathbf{f}_e,$$

where $\mathbf{J}^T = \mathbf{J}_a = \mathbf{\Phi}^T\mathbf{X}^T\mathbf{D}$, $\mathbf{M} = \mathbf{M}_a = \mathbf{\Phi}^T\mathbf{X}^T\mathbf{M}_q\mathbf{X}\mathbf{\Phi}$, $\mathbf{C} = \mathbf{C}_a = \mathbf{\Phi}^T\mathbf{X}^T\mathbf{b} + \mathbf{\Phi}^T\mathbf{X}^T\mathbf{M}_q\dot{\mathbf{X}}\mathbf{\Phi}\dot{\mathbf{q}}$ and the other symbols carry the same definitions as given before.

8.2. Free-Floating Manipulator

A free-floating manipulator system configuration is achieved when the base (spacecraft) is not controlled, i.e., when no external forces are acting on the system. For this case, only the base force is required to be set to zero, i.e., $\mathbf{f}_b = \mathbf{0}$.

Hence, now the dynamics of a free-floating manipulator system can be described by the following equations:

$$\mathbf{T}_a - \mathbf{J}^T \mathbf{f}_e = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C},$$

$$\dot{\mathbf{V}}^e = \mathbf{W}\dot{\mathbf{V}}^o + \dot{\mathbf{W}}\mathbf{V}^o = \mathbf{J}_b\dot{\mathbf{V}}_b + \mathbf{J}_q\ddot{\mathbf{q}} + \dot{\mathbf{J}}_b\mathbf{V}_b + \dot{\mathbf{J}}_q\dot{\mathbf{q}},$$

$$\mathbf{M}_o\dot{\mathbf{V}}^o + \mathbf{b}_o = \mathbf{W}^T\mathbf{f}_e\,.$$

All other symbols carry the same definitions as given before.

The end-effector force $\mathbf{f}_e$ occurs when the manipulator collides or captures an object in space and, thereafter, manipulates the captured object.

8.3. Free-Flying Manipulator

A free-flying manipulator system configuration is achieved when the base (spacecraft) is explicitly controlled by applying an external force by means of a thruster on the base. In this case, the dynamics can be represented by the systems of equation as given in the original development. Hence, the free-flying system dynamics can be represented by

$$\hat{\mathbf{T}}_{\bar{o}}\mathbf{J}^T \mathbf{f}_e = \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C},$$

$$\dot{\mathbf{V}}^e = \mathbf{W}\dot{\mathbf{V}}^o + \dot{\mathbf{W}}\mathbf{V}^o = \mathbf{J}_b\dot{\mathbf{V}}_b + \mathbf{J}_q\ddot{\mathbf{q}} + \dot{\mathbf{J}}_b\mathbf{V}_b + \dot{\mathbf{J}}_q\dot{\mathbf{q}},$$

$$\mathbf{M}_o\dot{\mathbf{V}}^o + \mathbf{b}_o = \mathbf{W}^T\mathbf{f}_e\,.$$

All the symbols carry the usual definitions as described before and the base force $\mathbf{f}_b$ is the active force applied by the thrusters. It is important to observe with reference to the base constraint force in flexible-base systems, where it is a passive force (Appendix C).

For a noncooperative, unconstrained manipulator system, the object dynamics is absent, and the end-effector force $\mathbf{f}_e$ can be set to zero. All the case formulations discussed above are explicitly defined for multiple arm systems. Single arm manipulators can be dealt with by setting the number-of-manipulators variable to one.

9. COMPUTATION OF MANIPULATOR MOTION

The algorithm for the motion simulation, with the unified system dynamics model as described above, can be described by the following series of steps:

Table I. Dynamic simulation algorithm for a unified systems model

Step 1:	Compute $\dot{\mathbf{V}}^e_{\text{open}}$ and model parameters
Step 2:	Solve for $\dot{\mathbf{V}}^o$
	$\dot{\mathbf{V}}^o = (\mathbf{M}_o^{-1} + \mathbf{W}^T\mathbf{H}^{-1}\mathbf{W})^{-1}\{\mathbf{H}^{-1}(\dot{\mathbf{V}}^e_{\text{open}} - \dot{\mathbf{W}}\mathbf{V}^o) - \mathbf{b}_o\}$
	where
	$\mathbf{H} = \mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T + \mathbf{J}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T(\mathbf{E}_{6nm} - \mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)\mathbf{D}$
Step 3:	Solve for $\mathbf{f}_e$
	$\mathbf{f}_e = \mathbf{H}^{-1}(\dot{\mathbf{V}}^e_{\text{open}} - \mathbf{W}\dot{\mathbf{V}}^o - \dot{\mathbf{W}}\mathbf{V}^o)$
Step 4:	Solve for $\ddot{\mathbf{q}}$
	$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\hat{\mathbf{T}} - \mathbf{J}^T\mathbf{f}_e - \mathbf{C})$
	Solve for $\dot{\mathbf{V}}_b$
	$\dot{\mathbf{V}}_b = \mathbf{M}_b^{-1}\{(\mathbf{E}_6 - \mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1})(\mathbf{f}_b - \mathbf{b}_b) - \mathbf{X}_b^T(\mathbf{E}_{6nm} - \mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)\mathbf{D}\mathbf{f}_e \quad - \mathbf{X}_b^T\mathbf{G}\mathbf{X}\boldsymbol{\Phi}\ddot{\mathbf{q}} - \mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}_b\mathbf{V}_b + \mathbf{X}_b^T\mathbf{G}\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{b} - \mathbf{X}_b^T\mathbf{G}\dot{\mathbf{X}}\boldsymbol{\Phi}\dot{\mathbf{q}}\mathbf{X}_b^T\mathbf{b}\}$
Step 5:	Use any numerical integration method to integrate joint and base accelerations to obtain the next state position and rates.

(i) solution of the unconstrained open-chain system,
(ii) object acceleration calculation,
(iii) end-effector force vector calculation,
(iv) calculation of the constrained closed-chain joint accelerations,
(v) calculations of base acceleration, and
(vi) integration of the joint rates and base acceleration to compute the next states.

In step 1, $\dot{\mathbf{V}}^e_{\text{open}}$ is calculated assuming zero end-effector forces. Then, in step 2, object acceleration is calculated using Eq. (45). Step 3 uses this acceleration to calculate the end-effector forces exerted on the object by the manipulators. With the given tip forces $\mathbf{f}_e$, the closed chain accelerations are calculated in step 4. Then, base acceleration is calculated in step 5. Finally, a fourth order Runge–Kutta integration is used to calculate the next state position and rates for all the bodies in the system. This algorithm, together with necessary computations, is summarized in Table I.

In the above implementation, the major issue to be considered is the computation of model parameters. This can be represented by the following steps:

(1) Calculation of transformation matrix $\mathbf{X}$ from the knowledge of corresponding rotation ma-

trices and link lengths. Transformation matrices between two nonconsecutive link frames can be calculated as follows:

$$ {}^{n}_{i}\mathbf{X} = {}^{n}_{n-1}\mathbf{X}\, {}^{n-1}_{i}\mathbf{X}. $$

(2) Calculation of the matrices $\mathbf{X}_b$, $\mathbf{B}$, and $\mathbf{D}$ can be performed by using the transformation matrices determined in step 1.

(3) Calculation of the derivative of $\mathbf{X}$, $\mathbf{X}_b$, $\mathbf{B}$ and $\mathbf{D}$.

The derivative of a transformation matrix ${}^{i+1}_{i}\mathbf{X}$ with respect to time is calculated as

$$ \frac{d}{dt}({}^{i+1}_{i}\mathbf{X}) = {}^{i+1}_{i}\mathbf{V}\hat{x}\, {}^{i+1}_{i}\mathbf{X}, $$

where ${}^{i+1}_{i}\mathbf{V}$ is the velocity of $(i+1)$th link frame with respect to the ith link frame represented in the $(i+1)$th link frame.

More specifically for rotational joints, this can be calculated simply as follows

$$ \frac{d}{dt}({}^{i+1}_{i}\mathbf{X}) = \begin{bmatrix} {}^{i+1}_{i}\Omega x & \mathbf{0} \\ \mathbf{0} & {}^{i+1}_{i}\Omega x \end{bmatrix} {}^{i+1}_{i}\mathbf{X}, $$

where ${}^{i+1}_{i}\Omega$ is the relative angular velocity between the ith and $(i+1)$th frame resolved in the $(i+1)$th frame. This can be defined as

$$ {}^{i+1}_{i}\Omega = {}^{i+1}_{i}\mathbf{R}\omega_i - \omega_{i+1} $$

Similarly, the derivative of the force transformation matrix ${}^{i}_{i+1}\mathbf{X}$ is given by

$$ \frac{d}{dt}({}^{i}_{i+1}\mathbf{X}) = \left(\frac{d}{dt}({}^{i}_{i+1}\mathbf{X})\right)^T = -{}^{i}_{i+1}\mathbf{X}\, {}^{i+1}_{i}\mathbf{V}\hat{x}. $$

In addition, for a rotational joint, this can be represented as

$$ \frac{d}{dt}({}^{i}_{i+1}\mathbf{X}) = -{}^{i}_{i+1}\mathbf{X} \begin{bmatrix} {}^{i+1}_{i}\Omega x & \mathbf{0} \\ \mathbf{0} & {}^{i+1}_{i}\Omega x \end{bmatrix}. $$

The derivative of the transformation matrices of nonconsecutive frames can be calculated as follows

$$ \frac{d}{dt}({}^{n}_{i}\mathbf{X}) = \frac{d}{dt}({}^{n}_{n-1}\mathbf{X})\, \frac{d}{dt}({}^{n-1}_{i}\mathbf{X}). $$

(4) Calculation of derivatives of $\mathbf{J}_b$ and $\mathbf{J}_q$ can be achieved using

$$ \frac{d}{dt}(\mathbf{J}_b) = \mathbf{B}\dot{\mathbf{X}}_b + \dot{\mathbf{B}}\mathbf{X}_b\,; $$

$$ \frac{d}{dt}(\mathbf{J}_q) = \mathbf{B}\dot{\mathbf{X}}\mathbf{\Phi} + \dot{\mathbf{B}}\mathbf{X}\mathbf{\Phi}. $$

(5) Calculation of $\mathbf{M}$, $\mathbf{J}^T$ and $\mathbf{C}$, together with the coefficients of $\mathbf{f}_b$, can be achieved as per the respective formulations in Eq. (28).

10. DISCUSSION

The inverse dynamics equation presented in Eq. (28) has a similar structure to that presented in Appendix A, Eq. (A9) for the work of Yoshida and Nenchev.[4] However, in Eq. (A9) the internal structures of the coefficients are not shown, which rather prohibits a direct comparison of Eqs. (28) and (A9). Nevertheless, in the initial work of Jain and Rodriguez,[7] the explicit internal structures are presented in Eqs. (A1) and (A2). Now, a comparison of the structures of $\mathbf{M}_a$ in both the cases shows a significant similarity, with the exception that the modal matrix in Eq. (28) is the entire system modal matrix $\mathbf{\Phi}$ whereas in (A2) it is the active joint modal matrix $\mathbf{\Phi}_a$. However, the structure of $\mathbf{M}_p$, $\mathbf{M}_{ap}$, $\mathbf{C}_p$, and $\mathbf{C}_a$ are significantly different in both the cases. This apparent difference in the results may be due to the fact that the models of Jain and Rodriguez, and Yoshida and Nenchev, are based on the underlying assumption of linear separability of the active and passive components. An intuitive analysis immediately focuses on this aspect, because the manipulator system is a highly nonlinear, coupled, dynamic system. Hence, the simple superposition philosophy of breaking the whole system into active and passive subsystems, and dealing with them separately, does not hold good for a nonlinear system. This might be the reason for the clear-cut difference in the internal structures of all the nonlinear term representations. However, it is noteworthy that the overall structure of the results of mapping of passive components onto the active ones through a dynamic coupling expression is true in all the cases.

The multiple degrees of freedom joints in a system can be all active or all passive or a mix of active and passive component degrees of freedom. The former two cases can be dealt with directly under the active or passive heading. However, a joint with a mixture of active and passive component degrees of freedom is considered in Jain and Rodriguez, with the assumption that this joint can be represented and modeled by an equivalent concatenation of active and passive joints. But, the model proposed here can very easily incorporate this type of mixed active and passive joint into the modal matrix $\mathbf{\Phi}$. This is the major advantage of modeling multiple degrees of freedom joints by a modal matrix.

11. CONCLUSIONS

This paper has described a unified approach to dynamic modeling of various manipulator configurations. The model has been developed using basic principles of mechanics. Spatial operator algebra has been used to describe the kinematic and dynamic behavior. Starting with very basic spatial operators, more complex spatial operators have been developed to describe the dynamical behavior.

After establishing the generality of the free base to represent various manipulator configurations, the model formulation has been discussed in the context of space robots. Finally, the results for a space robot have been extended to represent any underactuated manipulator configuration. A thorough analysis of all the kinematic and dynamic characteristics has been carried out during the course of formulating the model. A complete physical interpretation of the final equations of motion has been presented. The potential of SOA for gaining insights into the complex formulations has been strongly justified. The concept of open architecture for the simulation of close-chain mechanisms has also been emphasized. Keeping in view the complexity of the dynamics terms, the computer implementation of all the complex spatial operators has been described in detail. In addition to this, it has been shown that the linear separability assumptions for deriving unified models of the underactuated systems may not be valid in general. In contrast to this, the unified formulations discussed in this paper do not include any such assumptions, rather it has been derived from the first principles of mechanics.

APPENDIX A: UNIFIED DYNAMIC MODEL BASED ON LINEAR SEPARABILITY CONCEPT

The initial work of Jain and Rodriguez,[7] to model underactuated manipulators, serves as the major basis of the unified model described by Yoshida and Nenchev.[4] The basic concept of Jain and Rodriguez was that any underactuated manipulator system could equivalently be represented by one active and one passive arm. The active arm is a manipulator system assumed to be consisting of only active joints by freezing all the passive joints. Similarly, the passive arm results by freezing all the active joints.

Now, consider an underactuated manipulator system with $n(=n_a+n_p)$ joints and d_{ap} degrees of freedom with d_a the degrees of freedom of active joints and d_p the degrees of freedom of all passive joints. Here n_a and n_p denote the number of active and passive joints. Then define $\dot{\mathbf{q}}_a \in \mathbb{R}^{d_a}$, $\mathbf{T}_a \in \mathbb{R}^{d_a}$, and $\mathbf{\Phi}_a \in \mathbb{R}^{6n \times d_a}$ as the active joint velocities, torques and joint map matrix for the active arm. Further, define these quantities for the passive arm as $\dot{\mathbf{q}}_p \in \mathbb{R}^{d_p}$, $\mathbf{T}_p \in \mathbb{R}^{d_p}$ and $\mathbf{\Phi}_p \in \mathbb{R}^{6n \times d_p}$. Then, $\dot{\mathbf{q}}_a / \dot{\mathbf{q}}_p$, $\mathbf{T}_a / \mathbf{T}_p$ and $\mathbf{\Phi}_a / \mathbf{\Phi}_p$ are the decompositions of $\dot{\mathbf{q}}$, $\mathbf{T}$, and $\mathbf{\Phi}$, respectively. The equations of such system is expressed as

$$\begin{bmatrix} \mathbf{M}_a & \mathbf{M}_{ap} \\ \mathbf{M}_{ap}^T & \mathbf{M}_p \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_a \\ \ddot{\mathbf{q}}_p \end{bmatrix} + \begin{bmatrix} \mathbf{C}_a \\ \mathbf{C}_p \end{bmatrix} = \begin{bmatrix} \mathbf{T}_p \\ \mathbf{T}_a \end{bmatrix}, \tag{A1}$$

where $\mathbf{M}_a \in \mathbb{R}^{d_a \times d_a}$, $\mathbf{M}_p \in \mathbb{R}^{d_p \times d_p}$, and $\mathbf{M}_{ap} \in \mathbb{R}^{d_a \times d_p}$ are the mass matrices of active arm, passive arm and their interactions, respectively; and $\mathbf{C}_a$ and $\mathbf{C}_p$ are the active and passive coriolis and centrifugal forces of the active and passive arms, respectively. All these matrices are defined as follows:

$$\mathbf{M}_a = \mathbf{\Phi}_a^T \mathbf{X}^T \mathbf{M}_q \mathbf{X} \mathbf{\Phi}_a\,, \tag{A2a}$$

$$\mathbf{M}_p = \mathbf{\Phi}_p^T \mathbf{X}^T \mathbf{M}_q \mathbf{X} \mathbf{\Phi}_p\,, \tag{A2b}$$

$$\mathbf{M}_{ap} = \mathbf{\Phi}_a^T \mathbf{X}^T \mathbf{M}_q \mathbf{X} \mathbf{\Phi}_p\,, \tag{A2c}$$

$$\mathbf{C}_a = \mathbf{\Phi}_a^T \mathbf{X}^T (\mathbf{b} + \mathbf{M}\mathbf{X}\mathbf{a}), \tag{A2d}$$

$$\mathbf{C}_p = \mathbf{\Phi}_p^T \mathbf{X}^T (\mathbf{b} + \mathbf{M}\mathbf{X}\mathbf{a}). \tag{A2e}$$

Here, all the terms are as defined in the main text except $\mathbf{a}$, which defines the bias terms in the link acceleration.

Yoshida and Nenchev[4] extended these concepts to include end-effector forces and expounded the generality features of the model for a wide variety of manipulator configurations. The equations of motion with end-effector forces $\mathbf{f}_e$ can be represented as

$$\begin{bmatrix} \mathbf{M}_a & \mathbf{M}_{ap} \\ \mathbf{M}_{ap}^T & \mathbf{M}_p \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_a \\ \ddot{\mathbf{q}}_p \end{bmatrix} + \begin{bmatrix} \mathbf{C}_a \\ \mathbf{C}_p \end{bmatrix} = \begin{bmatrix} \mathbf{T}_p \\ \mathbf{T}_a \end{bmatrix} + \begin{bmatrix} \mathbf{J}_p^T \\ \mathbf{J}_a^T \end{bmatrix} \mathbf{f}_e , \tag{A3}$$

where $\mathbf{J}_a$ and $\mathbf{J}_p$ are the decompositions of the Jacobian $\mathbf{J}$ that maps $\mathbf{f}_e$ onto the active and passive joints.

The end-effector velocity is given by

$$\mathbf{V}^e = \mathbf{J}_a \dot{\mathbf{q}}_a + \mathbf{J}_p \dot{\mathbf{q}}_p . \tag{A4}$$

Then, the end-effector acceleration is expressed as

$$\dot{\mathbf{V}}^e = \mathbf{J}_a \ddot{\mathbf{q}}_a + \dot{\mathbf{J}}_a \dot{\mathbf{q}}_a + \mathbf{J}_p \ddot{\mathbf{q}}_p + \dot{\mathbf{J}}_p \dot{\mathbf{q}}_p \tag{A5}$$

Now, Eq. (A3) can be represented by the following equations:

$$\mathbf{M}_a \ddot{\mathbf{q}}_a + \mathbf{M}_{ap}^T \ddot{\mathbf{q}}_p + \mathbf{C}_a = \mathbf{T}_a + \mathbf{J}_a^T \mathbf{f}_e , \tag{A6}$$

$$\mathbf{M}_p \ddot{\mathbf{q}}_p + \mathbf{M}_{ap}^T \ddot{\mathbf{q}}_a + \mathbf{C}_p = \mathbf{T}_p + \mathbf{J}_p^T \mathbf{f}_e . \tag{A7}$$

Then, the passive co-ordinate acceleration vector $\ddot{\mathbf{q}}_p$ from Eq. (A7) can be written as

$$\ddot{\mathbf{q}}_p = -\mathbf{M}_p^{-1} (\mathbf{M}_{ap}^T \ddot{\mathbf{q}}_a + \mathbf{C}_p - \mathbf{T}_p - \mathbf{J}_p^T \mathbf{f}_e). \tag{A8}$$

Substitution of Eq. (A8) into (A6) yields

$$\mathbf{M} \ddot{\mathbf{q}}_a + \mathbf{C} = \hat{\mathbf{T}} + \mathbf{J}^T \mathbf{f}_c , \tag{A9}$$

where

$$\mathbf{M} = \mathbf{M}_a - \mathbf{M}_{ap}^T \mathbf{M}_p^{-1} \mathbf{M}_{ap} ,$$

$$\mathbf{J} = \mathbf{J}_a - \mathbf{J}_p \mathbf{M}_p^{-1} \mathbf{M}_{ap} ,$$

$$\mathbf{C} = \mathbf{C}_a - \mathbf{M}_{ap}^T \mathbf{M}_p^{-1} \mathbf{C}_p ,$$

$$\hat{\mathbf{T}} = \mathbf{T}_a - \mathbf{M}_{ap}^T \mathbf{M}_p^{-1} \mathbf{T}_p .$$

The passive component transformation factor $\mathbf{M}_{ap}^T \mathbf{M}_p^{-1}$ is known as the *dynamic coupling* expression.

Now, the end-effector acceleration in Eq. (A5) by using Eq. (A8), yields

$$\begin{aligned} \dot{\mathbf{V}}^e &= (\mathbf{J}_a - \mathbf{J}_p \mathbf{M}_p^{-1} \mathbf{M}_{ap}) \ddot{\mathbf{q}}_a + \dot{\mathbf{J}}_a \dot{\mathbf{q}}_a + \dot{\mathbf{J}}_p \dot{\mathbf{q}}_p \\ &\quad - \mathbf{J}_p \mathbf{M}_p^{-1} (\mathbf{C}_p - \mathbf{T}_p - \mathbf{J}_p^T \mathbf{f}_e) \\ &= \mathbf{J} \ddot{\mathbf{q}}_a + \ddot{\zeta}, \end{aligned} \tag{A10}$$

where the nonlinear acceleration $\ddot{\zeta}$ can be computed from a model or measurements.

APPENDIX B: DERIVATION OF THE COEFFICIENTS IN THE TEXT

B.1. Coefficients of Eq. (28)

As shown in the text, the inverse dynamics equation was obtained by multiplying $\mathbf{\Phi}^T$ to both sides of Eq. (27):

$$\begin{aligned} &\mathbf{\Phi}^T \mathbf{f} - \mathbf{\Phi}^T \mathbf{X}^T \{\mathbf{M}_q - \mathbf{M}_q \mathbf{X}_b (\mathbf{M}_b \\ &\quad + \mathbf{X}_b^T \mathbf{M}_q \mathbf{X}_b)^{-1} \mathbf{X}_b^T \mathbf{M}_q\} \mathbf{M}_q^{-1} \mathbf{D} \mathbf{f}_e \\ &= \mathbf{\Phi}^T \mathbf{X}^T \{\mathbf{M}_q - \mathbf{M}_q \mathbf{X}_b (\mathbf{M}_b \\ &\quad + \mathbf{X}_b^T \mathbf{M}_q \mathbf{X}_b)^{-1} \mathbf{X}_b^T \mathbf{M}_q\} \{\mathbf{X} \mathbf{\Phi} \ddot{\mathbf{q}} + \dot{\mathbf{X}}_b \mathbf{V}_b \\ &\quad + \mathbf{M}_q^{-1} \mathbf{b} + \mathbf{X}_b \mathbf{M}_b^{-1} (\mathbf{f}_b - \mathbf{b}_b) + \dot{\mathbf{X}} \mathbf{\Phi} \dot{\mathbf{q}}\}. \end{aligned} \tag{B1}$$

Now, Eq. (B1) produces the final inverse dynamics equation:

$$\hat{\mathbf{T}} - \mathbf{J}^T \mathbf{f}_e = \mathbf{M} \ddot{\mathbf{q}} + \mathbf{C}. \tag{B2}$$

Then, equating term by term all the terms of Eqs. (B1) and (B2) yields

$$\mathbf{M} = \mathbf{\Phi}^T \mathbf{X}^T \{\mathbf{M}_q - \mathbf{M}_q \mathbf{X}_b (\mathbf{M}_b + \mathbf{X}_b^T \mathbf{M}_q \mathbf{X}_b)^{-1} \mathbf{X}_b^T \mathbf{M}_q\} \mathbf{X} \mathbf{\Phi}, \tag{B3}$$

$$\mathbf{J}^T = \mathbf{\Phi}^T \mathbf{X}^T \{\mathbf{M}_q - \mathbf{M}_q \mathbf{X}_b (\mathbf{M}_b + \mathbf{X}_b^T \mathbf{M}_q \mathbf{X}_b)^{-1} \mathbf{X}_b^T \mathbf{M}_q\} \mathbf{M}_q^{-1} \mathbf{D}, \tag{B4}$$

$$\begin{aligned} \mathbf{C} &= \mathbf{\Phi}^T \mathbf{X}^T \{\mathbf{M}_q - \mathbf{M}_q \mathbf{X}_b (\mathbf{M}_b + \mathbf{X}_b^T \mathbf{M}_q \mathbf{X}_b)^{-1} \mathbf{X}_b^T \mathbf{M}_q\} \{\dot{\mathbf{X}}_b \mathbf{V}_b \\ &\quad + \mathbf{M}_q^{-1} \mathbf{b} - \mathbf{X}_b \mathbf{M}_b^{-1} \mathbf{b}_b + \dot{\mathbf{X}} \mathbf{\Phi} \dot{\mathbf{q}}\}, \end{aligned} \tag{B5}$$

$$\begin{aligned} \hat{\mathbf{T}} &= \mathbf{\Phi}^T \mathbf{f} - \mathbf{\Phi}^T \mathbf{X}^T \{\mathbf{M}_q - \mathbf{M}_q \mathbf{X}_b (\mathbf{M}_b \\ &\quad + \mathbf{X}_b^T \mathbf{M}_q \mathbf{X}_b)^{-1} \mathbf{X}_b^T \mathbf{M}_q\} \mathbf{X}_b \mathbf{M}_b^{-1} \mathbf{f}_b . \end{aligned} \tag{B6}$$

The following lemma proves the expressions for the coefficients of Eq. (B2) used in the text.

Lemma B1: Prove that

$$\mathbf{M} = \mathbf{M}_a - \mathbf{M}_{ap}^T \mathbf{M}_p^{-1} \mathbf{M}_{ap} , \tag{B7}$$

$$\mathbf{J}^T = \mathbf{J}_a - \mathbf{M}_{ap}^T \mathbf{M}_p^{-1} \mathbf{J}_p , \tag{B8}$$

$$\mathbf{C}=\mathbf{C}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{C}_p\,, \quad \text{(B9)}$$

$$\hat{\mathbf{T}}=\mathbf{T}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{T}_p\,, \quad \text{(B10)}$$

where $\mathbf{M}_a=\mathbf{\Phi}^T\mathbf{X}^T\mathbf{M}_q\mathbf{X}\mathbf{\Phi}$, $\mathbf{M}_{ap}^T=\mathbf{\Phi}^T\mathbf{X}^T\mathbf{M}_q\mathbf{X}_b$, $\mathbf{M}_p=\mathbf{M}_b+\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b$, and $\mathbf{M}_{ap}=\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}\mathbf{\Phi}$, $\mathbf{J}_a=\mathbf{\Phi}^T\mathbf{X}^T\mathbf{D}$, $\mathbf{J}_p=\mathbf{X}_b^T\mathbf{D}$, $\mathbf{C}_a=\mathbf{\Phi}^T\mathbf{X}^T\mathbf{b}+\mathbf{\Phi}^T\mathbf{X}^T\mathbf{M}_q\dot{\mathbf{X}}\mathbf{\Phi}\dot{\mathbf{q}}$, $\mathbf{C}_p=\mathbf{X}_b^T(\mathbf{b}+\mathbf{M}_q\dot{\mathbf{X}}\mathbf{\Phi}\dot{\mathbf{q}})-(\mathbf{M}_b\mathbf{X}_b^{-1}\dot{\mathbf{X}}_b\mathbf{V}_b-\mathbf{b}_b)$, $\mathbf{T}_p=\mathbf{f}_b$, and $\mathbf{T}=\mathbf{T}_a=\mathbf{\Phi}^T\mathbf{f}$.

Proof: The expressions for $\mathbf{M}$ in Eq. (B7) and $\mathbf{J}^T$ in Eq. (B8) can directly be obtained by expanding Eqs. (B3) and (B4), respectively.

Then, the expression for $\mathbf{C}$ can be obtained from the following subexpressions:

$$\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{M}_b\mathbf{X}_b^{-1}\dot{\mathbf{X}}_b\mathbf{V}_b =-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}(\mathbf{X}_b^T\mathbf{M}_q-\mathbf{M}_p\mathbf{X}_b^{-1})\dot{\mathbf{X}}_b\mathbf{V}_b \quad \text{(B11)}$$

$$-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{b}_b=\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}(\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b-\mathbf{M}_p)\mathbf{M}_b^{-1}\mathbf{b}_b\,. \quad \text{(B12)}$$

Now, expand Eq. (B5), and use Eqs. (B11) and (B12) to obtain

$$\begin{aligned}\mathbf{C}&=\mathbf{C}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}(\mathbf{X}_b^T\mathbf{b}+\mathbf{X}_b^T\mathbf{M}_q\dot{\mathbf{X}}\mathbf{\Phi}\dot{\mathbf{q}}\\&\quad-\mathbf{M}_b\mathbf{X}_b^{-1}\dot{\mathbf{X}}_b\mathbf{V}_b+\mathbf{b}_b)\\&=\mathbf{C}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{C}_p\,.\end{aligned}$$

Then, the expression for $\hat{\mathbf{T}}$ in Eq. (B10) can be obtained from the following relationship:

$$\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{f}_b=\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}(\mathbf{M}_p-\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b)\mathbf{M}_b^{-1}\mathbf{f}_b\,. \quad \text{(B13)}$$

Now, use Eqs. (B13) in Eq. (B6) to get Eq. (B10).

B.2. Disturbance Jacobian Matrix

The following lemma proves the expressions for the disturbance Jacobian matrix given in the text.

Lemma B2: Prove the following relationship

$$\mathbf{J}_D=\mathbf{M}_p^{-1}\mathbf{M}_{ap}\,. \quad \text{(B14)}$$

Proof: The proof of $\mathbf{J}_D$ can be as given below:

From Eqs. (26) and (38) it is clear that

$$\mathbf{J}_D=\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{G}\mathbf{X}\mathbf{\Phi}, \quad \text{(B15)}$$

where $\mathbf{G}=(\mathbf{M}_q^{-1}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T)^{-1}=\mathbf{M}_q(\mathbf{E}+\mathbf{X}_b\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{M}_q)^{-1}$.

Now, expansion of Eq. (B15) can yield

$$\mathbf{J}_D=\mathbf{M}_b^{-1}\mathbf{M}_{ap}-\mathbf{M}_b^{-1}\mathbf{X}_b^T\mathbf{M}_q\mathbf{X}_b\mathbf{M}_p^{-1}\mathbf{M}_{ap}=\mathbf{M}_p^{-1}\mathbf{M}_{ap}\,.$$

APPENDIX C: FLEXIBLE MANIPULATOR SYSTEMS

C.1. Flexible-Base Manipulator Systems

Consider a flexible-base manipulator system, whose base is constrained by a flexible-beam or a spring and damper (visco-elastic) system. The end-effectors are also subject to external forces. This flexible-base manipulator system can be expressed by the following system of equations:

$$\hat{\mathbf{T}}-\mathbf{J}^T\mathbf{f}_e=\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}, \quad \text{(C1)}$$

$$\mathbf{V}^e=\mathbf{J}_b\mathbf{V}_b+\mathbf{J}_q\dot{\mathbf{q}}, \quad \text{(C2)}$$

$$\dot{\mathbf{V}}^e=\mathbf{W}\dot{\mathbf{V}}^o+\dot{\mathbf{W}}\mathbf{V}^o=\mathbf{J}_b\dot{\mathbf{V}}_b+\mathbf{J}_q\ddot{\mathbf{q}}+\dot{\mathbf{J}}_b\mathbf{V}_b+\dot{\mathbf{J}}_q\dot{\mathbf{q}}, \quad \text{(C3)}$$

$$\mathbf{M}_o\dot{\mathbf{V}}^o+\mathbf{b}_o=\mathbf{W}^T\mathbf{f}_e\,, \quad \text{(C4)}$$

where $\hat{\mathbf{T}}=\mathbf{T}_a-\mathbf{M}_{ap}^T\mathbf{M}_p^{-1}\mathbf{T}_p$, $\mathbf{T}_p=\mathbf{f}_b$ is the base constraint force.

Usually, in the constraint force $\mathbf{f}_b$ can be represented as

$$\mathbf{f}_b=-\delta_b\dot{\mathbf{x}}_b-s_b\Delta\mathbf{x}_b\,, \quad \text{(C5)}$$

where δ_b is the damping factor, s_b is the spring factor, and $\Delta\mathbf{x}_b$ is the base displacement from the equilibrium position due to its elasticity.

C.2. Flexible-Arm Manipulator Systems

An approximate modeling of a flexible-link can be obtained by a successive chain of a finite number of virtual elastic joints. This formulation satisfies to the features of an underactuated system. Thus, this allows the use of a unified model formulated in the text to represent a flexible-arm manipulator system. The equations of motion of a flexible-arm system can be represented as

$$\hat{\mathbf{T}}-\mathbf{J}^T\mathbf{f}_e=\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}, \quad \text{(C6)}$$

$$\mathbf{V}^e = \mathbf{J}_b \mathbf{V}_b + \mathbf{J}_q \dot{\mathbf{q}}, \qquad \text{(C7)}$$

$$\dot{\mathbf{V}}^e = \mathbf{W}\dot{\mathbf{V}}^o + \dot{\mathbf{W}}\mathbf{V}^o = \mathbf{J}_b \dot{\mathbf{V}}_b + \mathbf{J}_q \ddot{\mathbf{q}} + \dot{\mathbf{J}}_b \mathbf{V}_b + \dot{\mathbf{J}}_q \dot{\mathbf{q}}, \qquad \text{(C8)}$$

$$\mathbf{M}_o \dot{\mathbf{V}}^o + \mathbf{b}_o = \mathbf{W}^T \mathbf{f}_e\,, \qquad \text{(C9)}$$

where $\hat{\mathbf{T}} = \mathbf{T}_a - \mathbf{M}_{ap}^T \mathbf{M}_p^{-1} \mathbf{T}_p$, $\mathbf{T}_p = \mathbf{f}_b$ is the force required to deflect the flexible joints, and $\mathbf{V}_b$ represents the time rate of change of elastic deflections of the flexible links.

Now, the force of elastic deflection $\mathbf{f}_b$ can be represented in terms of the stiffness (s_b) and damping (δ_b) matrices as:

$$\mathbf{f}_b = -\delta_b \dot{\mathbf{x}}_b - s_b \Delta \mathbf{x}_b \qquad \text{(C10)}$$

where $\Delta \mathbf{x}_b$ is the elastic displacement from the equilibrium position due to its elasticity.

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