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Availability of a system with gamma life and exponential repair time under a perfect repair policy

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Abstract

Employing Fourier transform technique we determine the availability of a maintained system under continuous monitoring and with perfect repair policy. We obtain closed-form expressions for the availability when the system has gamma life distribution and the repair time is exponential. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

We consider systems which are maintained and repaired upon each failure. We study the availability of the system; that is, the probability that the system is in the functioning state at any specified time. Availability is a measure of performance of the maintained system and is an important aspect of reliability theory. For an excellent account on the subject see, for example, Høyland and Rausand (1994).

We assume that the system is under continuous monitoring. Each time the system fails a repair ensues and (when the repair is completed) the system is restored back to a level equivalent to a new system. This is known as the perfect repair model (see Barlow and Proschan, 1975). We also assume that the repair is not instantaneous, rather repair takes a random amount of time.

Let F be the distribution of time to failure of the system and let G denote the repair time distribution. Assume that F and G are independent and they have continuous densities f and g respectively, with support on $(0,\infty)$. We are interested in the system availability $A(t)$ at any specified time $t>0$. It is well known that

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the limiting availability is

$$A(\infty) := \lim_{t\to\infty} A(t) = \frac{\text{MTTF}}{\text{MTTF} + \text{MTTR}}, \tag{1.1}$$

where MTTF stands for mean time to failure and MTTR for mean time to repair. Some authors have taken (1.1) as the definition of availability. See Sen and Bhattacharya (1986).

In the literature attempts have been made to find $A(t)$ employing Laplace transform technique (see, for example, Barlow and Proschan, 1975; Høyland and Rausand, 1994). But problems arise in inverting the Laplace transform. Except in the case when the underlying distributions are exponential, this is a formidable task.

In this paper we show that $A(t)$, the exact availability of the maintained system at time $t > 0$, can be obtained by integrating the sum of residues of a complex-valued function which is analytic except at finite number of singularities.

In Section 2 we present the main result and in Section 3 we obtain exact expressions for $A(t)$ when the system life distribution is gamma and repair time is exponential.

2. Main result

Suppose that at time $t = 0$, a new system starts to function. It continues to function for a random time T_1 until the first failure occurs. A repair ensues immediately and the repair is completed in a random time D_1 when the system is brought back to a condition as good as new and it starts to function again. The process is repeated under exactly the same conditions. Assume that $T_1, T_2, \ldots$ are independent and identically distributed (IID) random variables representing the successive times to failure with distribution function F (with absolutely continuous density f); and $D_1, D_2, \ldots$ are IID random variables representing the corresponding repair times with distribution function G (with absolutely continuous density g). Assume also that the times to failure are independent of the repair times.

Let the state of the system be denoted by a random indicator process $X(t)$. We write $X(t) = 1$ if the system is functioning at time t and $X(t) = 0$ if it is under repair at time t. The availability at time t is defined as $A(t) = P[X(t) = 1]$, the probability that the system is in the functioning state at time t.

To state the main result, the following notations and definitions are needed. Let $\tilde{f}$ and $\tilde{g}$ be the Fourier transform of f and g, respectively. Recall that the Fourier transform of a function $y(t)$ on $(-\infty, \infty)$ is a complex valued function defined for any $s \in (-\infty, \infty)$ by

$$\tilde{y}(s) := \int_{-\infty}^{\infty} e^{ist} y(t)\, dt, \tag{2.1}$$

where $i = \sqrt{-1}$ denotes the imaginary unit. Define for $z \in \mathbb{C}$,

$$c_t(z) := e^{-itz} \tilde{f}(z) \frac{1 - \tilde{g}(z)}{1 - \tilde{f}(z)\tilde{g}(z)}. \tag{2.2}$$

Theorem 2.1. *Suppose that $c_t(z)$, as defined in (2.2), is analytic except for isolated singularities $z_1, \ldots, z_k$ in the lower half plane (LHP). Then the availability at time $t > 0$ of a system with arbitrary life distribution F (with absolutely continuous density f) and arbitrary repair time distribution G (with absolutely continuous density g) is given by*

$$A(t) = 1 - \int_0^t b(u)\, du,$$

where

$$b(t) = -\mathrm{i} \sum_{z_j \in \mathrm{LHP}} \mathrm{Res}_{z_j}\, c_t(z) \tag{2.3}$$

with $\mathrm{Res}_{z_j}\, c_t(z)$ *being the contour integral of* $c_t(z)$ *along a counterclockwise circle with center* z_j *and radius small enough to include no other singularity.*

Proof. Let $B(t) = 1 - A(t) = P[X(t) = 0]$ denote the probability that the system is in the repair state at time t. Under the perfect repair model, we can express $B(t)$ satisfying the following integral equation:

$$\begin{aligned} B(t) &= P[X(t) = 0] \\ &= P[T_1 \leqslant t < T_1 + D_1] + P[X_t = 0 | T_1 + D_1 \leqslant t] \\ &= \{P[T_1 \leqslant t] - P[T_1 + D_1 \leqslant t]\} + \int_0^t P[X_t = 0 | T_1 + D_1 = x]\, \mathrm{d}H(x) \\ &= F(t) - H(t) + \int_0^t B(t - x)\, h(x)\, \mathrm{d}x, \end{aligned} \tag{2.4}$$

where $H(h)$ denotes the distribution (density) function of $T_1 + D_1$. Clearly, h is the convolution of f and g defined by

$$h(x) := \int_0^x f(x - u) g(u)\, \mathrm{d}u, \tag{2.5}$$

so that its Fourier transform is $\tilde{h}(s) = \tilde{f}(s)\tilde{g}(s)$.

Notice that $h(t) = 0$ for $t \in (-\infty, 0)$; also we can define $B(t) = 0$ for $t \in (-\infty, 0]$. Hence, the domain of the integral in (2.4) may be extended over the entire real line. Taking the derivative with respect to t in (2.4), and writing $b(t) = B'(t)$, we have

$$b(t) = f(t) - h(t) + \int_{-\infty}^{\infty} b(t - x) h(x)\, \mathrm{d}x. \tag{2.6}$$

By taking Fourier transforms on both sides of (2.6), we get

$$\tilde{b}(s) = \tilde{f}(s) - \tilde{h}(s) + \tilde{b}(s)\tilde{h}(s)$$

whence

$$\tilde{b}(s) = \frac{\tilde{f}(s) - \tilde{h}(s)}{1 - \tilde{h}(s)} = \tilde{f}(s) \frac{1 - \tilde{g}(s)}{1 - \tilde{f}(s)\tilde{g}(s)}. \tag{2.7}$$

Next, by the inversion formula, we recover $b(t)$ as

$$\begin{aligned} b(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ts} \tilde{b}(s)\, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ts} \tilde{f}(s) \frac{1 - \tilde{g}(s)}{1 - \tilde{f}(s)\tilde{g}(s)}\, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} c(s)\, \mathrm{d}s. \end{aligned} \tag{2.8}$$

Since $c_t(z)$ is analytic in a domain that includes the real axis and all of the lower half plane (LHP), except at isolated singularities $z_1, \ldots, z_k$; applying Theorem 33 of Kaplan (1984, p. 631) and Cauchy's Residue Theorem (see, for example, Kaplan, 1984, p. 619) we can write

$$\int_{-\infty}^{\infty} c(s)\,\mathrm{d}s = -2\pi\mathrm{i}\sum_{z_j} \mathrm{Res}_{z_j}\, c_t(z). \tag{2.9}$$

This completes the proof of the theorem. □

Theorem 2.1 provides us with an alternative technique for finding the availability of a maintained system in a general set up. Thus the problem of finding $A(t)$ amounts to solving for the singularities of some analytic function, which is likely to be easier than inverting a Laplace transform. In the next section we illustrate the above method of finding $A(t)$ through some examples.

3. Some examples

Example 1. Suppose that $T \sim$ Exponential(α) and $D \sim$ Exponential(β). This case has been worked out in the literature (see Høyland and Rausand, 1994, p. 309) using Laplace transform technique. Here we demonstrate that Fourier transform technique also yields the same result. Note that in this case

$$f(t) = \alpha \mathrm{e}^{-\alpha t}, \quad t > 0,$$

$$\tilde{f}(s) = (1 - \mathrm{i}s/\alpha)^{-1}, \quad -\infty < s < \infty,$$

$$g(t) = \beta \mathrm{e}^{-\beta t}, \quad t > 0,$$

$$\tilde{g}(s) = (1 - \mathrm{i}s/\beta)^{-1}, \quad -\infty < s < \infty,$$

$$\tilde{b}(s) = (1 + \beta/\alpha - \mathrm{i}s/\alpha)^{-1}, \quad -\infty < s < \infty,$$

$$c_t(z) = \mathrm{e}^{-\mathrm{i}tz}\,(1 + \beta/\alpha - \mathrm{i}z/\alpha)^{-1}, \quad z \in \mathbb{C}.$$

Here $c_t(z)$ has only one singularity at $z_1 = -\mathrm{i}(\alpha + \beta)$, which is the solution to $0 = 1 + \beta/\alpha - \mathrm{i}z/\alpha$. Since $c_t(z)$ is a rational function, by the rules of residue calculus, $\mathrm{Res}_{z_1} c_t(z) = \mathrm{i}\alpha \mathrm{e}^{-\mathrm{i}tz_1}$. Now (2.3) yields

$$b(t) = \alpha \mathrm{e}^{-(\alpha+\beta)t} \tag{3.1}$$

and hence

$$A(t) = \frac{\beta}{\alpha + \beta} + \frac{\alpha}{\alpha + \beta}\mathrm{e}^{-(\alpha+\beta)t}, \quad t > 0. \tag{3.2}$$

We note from (3.2) that $A(\infty) = \beta/(\alpha + \beta)$ as expected. Specializing to the case when $\alpha = \beta$, we have

$$A(t) = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{e}^{-2\alpha t}, \quad t > 0. \tag{3.3}$$

Example 2. Suppose that $T \sim$ Gamma(p, α) and $D \sim$ Exponential(β). We consider the two cases (a) $\alpha = \beta$ and (b) $\alpha \neq \beta$ separately,

Case 2a. $\alpha = \beta$. Note that in this case

$$f(t) = \frac{\alpha^p}{\Gamma(p)}\mathrm{e}^{-\alpha t} t^{p-1}, \quad t > 0,$$

$$\tilde{f}(s) = (1 - \mathrm{i}s/\alpha)^{-p}, \quad -\infty < s < \infty,$$

$$g(t) = \alpha \mathrm{e}^{-\alpha t}, \quad t > 0,$$

$$\tilde{g}(s) = (1 - \mathrm{i}s/\alpha)^{-1}, \quad -\infty < s < \infty,$$

$$\tilde{b}(s) = \frac{-\mathrm{i}s/\alpha}{(1-\mathrm{i}s/\alpha)^{p+1}-1}, \quad -\infty < s < \infty,$$

$$c_t(z) = \mathrm{e}^{-\mathrm{i}tz}\frac{-\mathrm{i}z/\alpha}{(1-\mathrm{i}z/\alpha)^{p+1}-1}, \quad z \in \mathbb{C}.$$

Here $c_t(z)$ is of the form 0/0 at $z_0=0$. In fact, one can simplify the expression to avoid this difficulty since z appears as a factor in both the numerator and the denominator of $c_t(z)$. Thus, the residues of $c_t(z)$ need to be calculated only at the non-zero singularities. Note that $c_t(z)$ has p non-zero isolated singularities at $z_j = -\mathrm{i}\alpha(1-\theta_j)$, for $j=1,\dots,p$; where $\theta_0=1,\theta_1,\dots,\theta_p$ are the $(p+1)$th roots of 1. In fact, $\theta_j = [\mathrm{e}^{\mathrm{i}2\pi/(p+1)}]^j$. Also note that $z_1,\dots,z_p$ are in the LHP. Since $c_t(z)$ is a rational function, by the rules of residue calculus,

$$\begin{aligned}\mathrm{Res}_{z_j}\, c_t(z) &= \mathrm{e}^{-\mathrm{i}tz_j}\frac{(-\mathrm{i}z_j/\alpha)}{(-\mathrm{i}/\alpha)(p+1)(1-\mathrm{i}z_j/\alpha)^p}\\ &= -\mathrm{i}\frac{\alpha}{p+1}(1-\theta_j)\theta_j\mathrm{e}^{-\alpha(1-\theta_j)t}.\end{aligned}$$

Hence, (2.3) yields

$$b(t) = -\frac{\alpha}{p+1}\sum_{j=1}^{p}(1-\theta_j)\theta_j\mathrm{e}^{-\alpha(1-\theta_j)t}. \tag{3.4}$$

Next, using the fact that $1+\theta_1+\cdots+\theta_p=0$, we have

$$A(t) = \frac{p}{p+1} - \frac{1}{p+1}\sum_{j=1}^{p}\theta_j\mathrm{e}^{-\alpha(1-\theta_j)t}, \quad t>0. \tag{3.5}$$

Note from (3.5) that $A(\infty)=p/(p+1)$ which agrees with (1.1), since MTTF$=p/\alpha$ and MTTR$=1/\alpha$.

Specializing to the case when $p=1$, and hence $\theta_1=-1$, we get back the same result as in (3.3). For the case $p=2$, we have $\theta_1=\mathrm{e}^{\mathrm{i}2\pi/3}=(-1+\mathrm{i}\sqrt{3})/2$ and $\theta_2=\mathrm{e}^{-\mathrm{i}2\pi/3}=-(1+\mathrm{i}\sqrt{3})/2$; and so $z_1=-\alpha(\sqrt{3}+\mathrm{i}3)/2$ and $z_2=-\alpha(-\sqrt{3}+\mathrm{i}3)/2$ are in the LHP; and

$$\begin{aligned}A(t) &= \tfrac{2}{3} - \tfrac{2}{3}\mathrm{e}^{-3\alpha t/2}\cos(2\pi/3+\sqrt{3}\alpha t/2)\\ &= \tfrac{2}{3} + \tfrac{1}{3}\mathrm{e}^{-3\alpha t/2}[\cos(\sqrt{3}\alpha t/2)+\sqrt{3}\sin(\sqrt{3}\alpha t/2)].\end{aligned} \tag{3.6}$$

For the case $p=3$, we have $\theta_1=\mathrm{i}$, $\theta_2=-1$, $\theta_3=-\mathrm{i}$. Consequently, $z_1=-\alpha(1+\mathrm{i})$, $z_2=-\mathrm{i}2\alpha$, $z_3=\alpha(1-\mathrm{i})$ are in the LHP; and

$$A(t) = \tfrac{3}{4} + \tfrac{1}{4}\mathrm{e}^{-2\alpha t} + \tfrac{1}{2}\mathrm{e}^{-\alpha t}\sin(\alpha t). \tag{3.7}$$

Fig. 1 depicts the availability when $\alpha=\beta$ and $p=1,2,3$, given in (3.3), (3.6) and (3.7), respectively. As anticipated, the availability increases as p increases (that is, as the system lifetime becomes stochastically larger).

Case 2b. $\alpha \neq \beta$. Note that in this case

$$\tilde{f}(s) = (1-\mathrm{i}s/\alpha)^{-p}, \quad -\infty < s < \infty,$$

$$\tilde{g}(s) = (1-\mathrm{i}s/\beta)^{-1}, \quad -\infty < s < \infty,$$

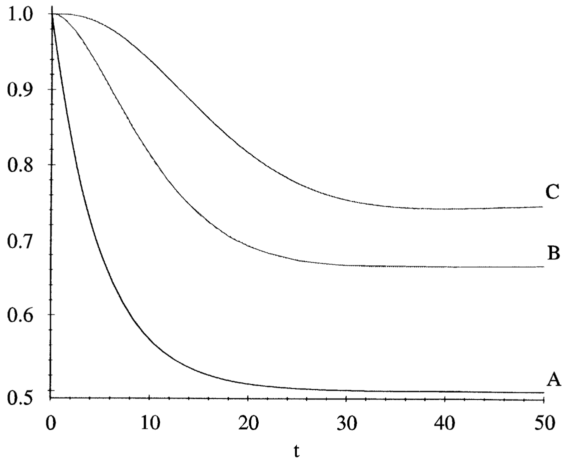

Fig. 1. System availability when $\alpha=\beta=0.1$ and (A) $p=1$, (B) $p=2$ and (C) $p=3$.

$$\tilde{b}(s)=\frac{-\mathrm{i}s/\beta}{(1-\mathrm{i}s/\alpha)^p(1-\mathrm{i}s/\beta)-1},\quad -\infty<s<\infty,$$

$$c_t(z)=\mathrm{e}^{-\mathrm{i}tz}\frac{-\mathrm{i}z/\beta}{(1-\mathrm{i}z/\alpha)^p(1-\mathrm{i}z/\beta)-1},\quad z\in\mathbb{C}.$$

Here also $c_t(z)$ is of the form 0/0 at $z_0=0$. But, as in Case 2a, the residues of $c_t(z)$ need to be calculated only at the non-zero singularities. Finding these non-zero singularities require finding the zeroes of a polynomial (over $\mathbb{C}$) of degree p. Example 1 above already exhibited the method for $p=1$. Below we illustrate the method for $p=2$.

Assume $p=2$. In this case, $c_t(z)$ reduces to

$$c_t(z)=-\alpha^2\mathrm{e}^{-\mathrm{i}tz}[z^2+\mathrm{i}(2\alpha+\beta)z-\alpha(\alpha+2\beta)]^{-1}.$$

Hence, the two non-zero isolated singularities of $c_t(z)$ are

$$z_1=\tfrac{1}{2}[-\mathrm{i}(2\alpha+\beta)+\sqrt{\beta(4\alpha-\beta)}]:=-\mathrm{i}(l-k),$$
$$z_2=\tfrac{1}{2}[-\mathrm{i}(2\alpha+\beta)-\sqrt{\beta(4\alpha-\beta)}]:=-\mathrm{i}(l+k), \tag{3.8}$$

where

$$l=\alpha+\beta/2\quad\text{and}\quad k=(1/2)\sqrt{\beta(4\alpha-\beta)}.$$

Note that z_1,z_2 are in the LHP. Also note that $z_1=z_2$ if and only if $k=0$. The calculations for the residues differ depending on the value of k. We consider the following cases:

Case 2b(i). ($4\alpha\neq\beta$). Here $c_t(z)$ has two simple poles at $z_1\neq z_2$ as given in (3.8) above; and

$$\mathrm{Res}_{z_j}\,c_t(z)=\lim_{z\to z_j}(z-z_j)\frac{(-\alpha^2)\mathrm{e}^{-\mathrm{i}tz_j}}{(z-z_1)(z-z_2)}.$$

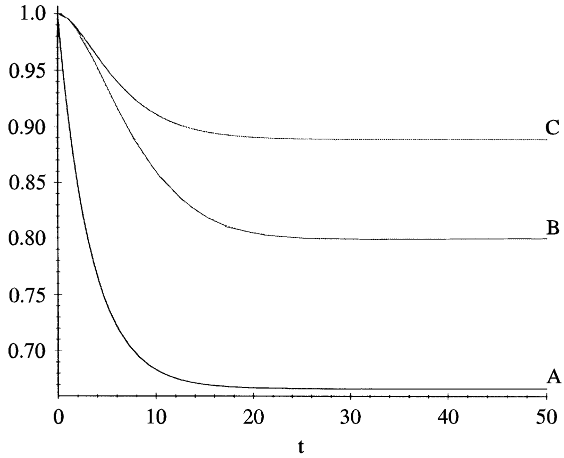

Fig. 2. System availability when (A) $\alpha = 0.1,\ \beta = 0.2,\ p = 1$, (B) $\alpha = 0.1,\ \beta = 0.2,\ p = 2$ and (C) $\alpha = 0.1,\ \beta = 0.4,\ p = 2$.

Hence, by (2.3) we have

$$b(t) = \alpha^2 \left[\frac{\mathrm{i}e^{-\mathrm{i}tz_1}}{z_1 - z_2} + \frac{\mathrm{i}e^{-\mathrm{i}tz_2}}{z_2 - z_1} \right] = \frac{\alpha^2}{k}\, e^{-lt} \sin(kt),$$

which is a real-valued function, since k is either purely real or purely imaginary and $\sin(\mathrm{i}y) = \mathrm{i}\sinh(y)$ for any real y. Hence, we have

$$A(t) = \frac{2\beta}{\alpha + 2\beta} + \frac{\alpha}{\alpha + 2\beta}\, e^{-lt} \left\{ \cos(kt) + \frac{l}{k}\sin(kt) \right\}, \quad t > 0. \tag{3.9}$$

Note that $A(t)$ is also a real-valued function, since $\cos(\mathrm{i}y) = \cosh(y)$ for any real y. Also note that $A(\infty) = 2\beta/(\alpha+2\beta)$ which agrees with (1.1), since MTTF=$2/\alpha$ and MTTR=$1/\beta$. Furthermore, specializing to the case when $\alpha = \beta$, we get back the same result as in (3.6).

Case 2b(ii). ($4\alpha = \beta$). Here $c_t(z)$ has one double pole at $z_* = -\mathrm{i}3\alpha$; and by Cauchy's Integral Formula (see, Smith, 1974, p. 206)

$$\mathrm{Res}_{z_*} c_t(z) = -\alpha^2 \frac{\mathrm{d}}{\mathrm{d}z}(e^{-\mathrm{i}tz})|_{z=z_*} = \mathrm{i}\alpha^2 t e^{-\mathrm{i}tz_*}.$$

Hence, (2.3) yields

$$b(t) = \alpha^2 t\, e^{-3\alpha t}$$

and

$$A(t) = \tfrac{8}{9} + \tfrac{1}{9}(3\alpha t + 1)\, e^{-3\alpha t}, \quad t > 0 \tag{3.10}$$

which may also be obtained by letting $k \to 0$ in (3.9). Finally, note that $A(\infty) = 8/9$ which agrees with (1.1), since MTTF $= 2/\alpha$ and MTTR $= 1/\beta = 1/(4\alpha)$.

Fig. 2 depicts the availability when $\alpha \neq \beta$ and $p = 1$ or $p = 2$, given in (3.2), (3.9) and (3.10), respectively. Again, as anticipated, the availability increases as p increases or as β increases.

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