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ESTIMATION OF CHANGE-POINT WHEN HAZARD RATE CHANGES SHARPLY

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Abstract

The hazard rate model

$$\begin{aligned} r(t) &= \alpha \quad \text{if} \quad 0 \le t < \tau, \\ &= \beta \quad \text{if} \quad \tau \le t < \infty, \end{aligned}$$

arises quite commonly in mechanical or biological systems, which experience a high hazard rate early in their lifetime due to infant mortality and then a constant or steady hazard rate after the threshold time τ. In this paper, we use the total time on test transform approach to estimate the change point τ and study its properties. We also compare this estimator with available estimators using a simulation study.

1 Introduction

In studies on hazard rates in Survival Analysis and Reliability studies, it is often the case that initially the hazard rate is high and then after a rapid or abrupt fall, it stabilizes at a lower value. If a treatment is given to a patient and then "survival" up to a relapse or some other identified episode is studied, then the patients who "survive" the initial shock of

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a new treatment like chemotherapy will develop low hazard rates. A similar situation in reliability problems has led to the so-called "burn-in" techniques to screen out defective electrical or electronic items and thus improve performance of the remaining items.

We may use the hazard rate change-point model (due to Matthew's and Farewell (1982)) in these situations, given by

$$r(t) = \begin{cases} \alpha & \text{if } \ 0 \leq t \leq \tau \\ \beta & \text{if } \ t > \tau \quad \alpha, \beta, \tau > 0 \end{cases} . \tag{1}$$

Here $\alpha > \beta$ indicates that the hazard rate moves from a high value to a low value. This model can be used for situation where infant mortality is quite high.

Recently Ghosh, Joshi and Mukhopadhyay (1998) studied the problem of estimating the change point in the hazard rate model (1.1) under the case $\alpha > \beta$. It is well known that the likelihood corresponding to a sample from model (1.1) is unbounded. Ghosh, Joshi and Mukhopadhyay (1998) proved the boundedness of the likelihood under the constraint $\alpha > \beta$. However, they also observed that even though the likelihood is bounded, it can still have a local maximum far to the right of τ, at or near $Max(X_1, X_2, \cdots, X_n)$, of magnitude comparable with the global maximum. Hence, they emphasized the need for restricting the likelihood estimates. They derive MMLE with an additional condition that $F(\tau) \leq 0.5$. They also compare their estimates with other existing estimates.

Most of the estimators available in the literature have been derived under the assumption that $\alpha > \beta$. However, there are situations in which the failure rate is low in the beginning and after some age, the failure rate stablizes at a higher value. For example, a manufactured product, where the infant mortality is low, is likely to have less failure rate in the beginning but after some age, fatigue sets in and the failure rate becomes high and remain the same for the rest of its life. In these situations, it is evident that $\alpha < \beta$. In this paper, we derive an estimator for τ which can be used even when $\alpha < \beta$.

In this paper, we derive the Total Time on Test (TTT) - Transform corresponding to the model (1.1) and derive estimator for τ on the basis of the TTT - plot using the properties of TTT transform in such a model and study its properties. In Section 2, we briefly discuss the TTT - transform corresponding to the model (1.1) and derive an estiamtor for τ on the basis of TTT - Plot. In Section 3, we study asymptotic properties of the estimator and in Section 4, we compare the estimator with other existing estimators using a simulation study.

2 Estimation of τ Using TTT-Transform

2.1 TTT-Transform

The TTT - Plot an empirical and scale invariant plot based on failure data, and the corresponding asymptotic curve, named the scaled TTT - Transform were introduced by Barlow and Campo (1975) and used for model identification purposes. Since then these tools have proven to be very useful in several applications within reliability. The TTT - Transform has also been found quite useful in theoretical applications such as looking for test statistics for particular purposes and to study their power. They are also useful in practical applications in analysis of ageing properties, maintenance optimization and also in design of experiments (See Deshpande and Suresh (1990) and Bergman and Klefsjo (1998)).

The scaled TTT - Transform of a life distribution F is defined as

$$\psi_F(u) = (1/\mu) \int_0^{h(u)} R(x)dx, \quad 0 \le u \le 1$$

where $R(t) = 1 - F(t)$ is the survival function, $\mu < \infty$ is the mean, and $h(t) = \text{inf. } \{x : F(x) \ge t\}, 0 < t \le 1, h(0) = 0$, is the inverse of the cdf F.

It is well known that $\Psi_G(u) = u, 0 \le u \le 1$ for the exponential distribution meaning that every exponential distribution $G(t) = 1 - \exp(-\lambda t), t \ge 0$ is transformed into the diagonal in the unit square independently of the value of the failure rate λ.

Suppose that we have a complete ordered sample $0 = t(0) \le t(1) \le \cdots \le t(n)$ from a life distribution F and corresponding TTT - statistics

$s_j = nt(1) + (n-1)(t(2) - t(1)) + \cdots + (n-j+1)(t(j) - t(j-1)), j = 1, \cdots, n$, (for convenience set $S_0 = 0$), then the piece wise linear graph $\Psi_n(t)$ obtained by joining the points $(j/n, u_j)$ where $u_j = S_j/S_n, j = 0, 1, \cdots, n$ is called the TTT - plot based on these observations. It may be easily see that the TTT-Plot is an empirical version of scaled TTT - transform. It is well known that TTT - plot converges with probability one and uniformly to the scaled TTT - transform (see Langberg, et. al. (1980)).

Now, consider the change - point hazard rate model given in (1.1). The corresponding survival function $R(t)$ is given by

$$R(t) = \begin{cases} \exp(-\alpha t), & 0 < t \leq \tau \\ \exp(-[\alpha\tau + \beta(t-\tau)]), & t \geq \tau. \end{cases}$$

It may be noted that the corresponding life distribution F is continuous with mean

$$\mu = p_0/\alpha + (1-p_0)/\beta < \infty,$$

and inverse of the cumulative distribution function given by

$$\begin{aligned} h(t) &= -ln(1-t)/\alpha \quad \text{for} \quad t < p_0 \\ &= \tau - [\alpha\tau + ln(1-t)]/\beta, \quad \text{for} \quad p_0 \leq t \leq 1 \end{aligned}$$

with

$$p_0 = F(\tau) = 1 - \exp(-\alpha\tau).$$

The TTT-Transform $\Psi_F(t)$ corresponding to the above model is given by

$$\Psi_F(t) = \begin{cases} b_0 t, & 0 < t < p_0 \\ a_1 + b_1 t, & p_0 \leq t \leq 1 \end{cases} \tag{1}$$

where $b_0 = \frac{1}{\mu\alpha}, \quad a_1 = \frac{p_0}{\mu\beta}, \quad b_1 = \frac{1}{\mu\beta}$.

Thus the TTT-Transform corresponding to model (1.1) is a two phase linear function with change point p_0. It may be noted that $\alpha = \beta$ or $\tau = 0$ is equivalent to $b_1 = b_0$ (and hence $a_1 = 0$ or $p_0 = 0$). The TTT - Transform for different values of the parameters are plotted in Figures 2.1 and 2.2.

Figure 2.1 : TTT-plot with alpha=3,beta=1,tau=0.15

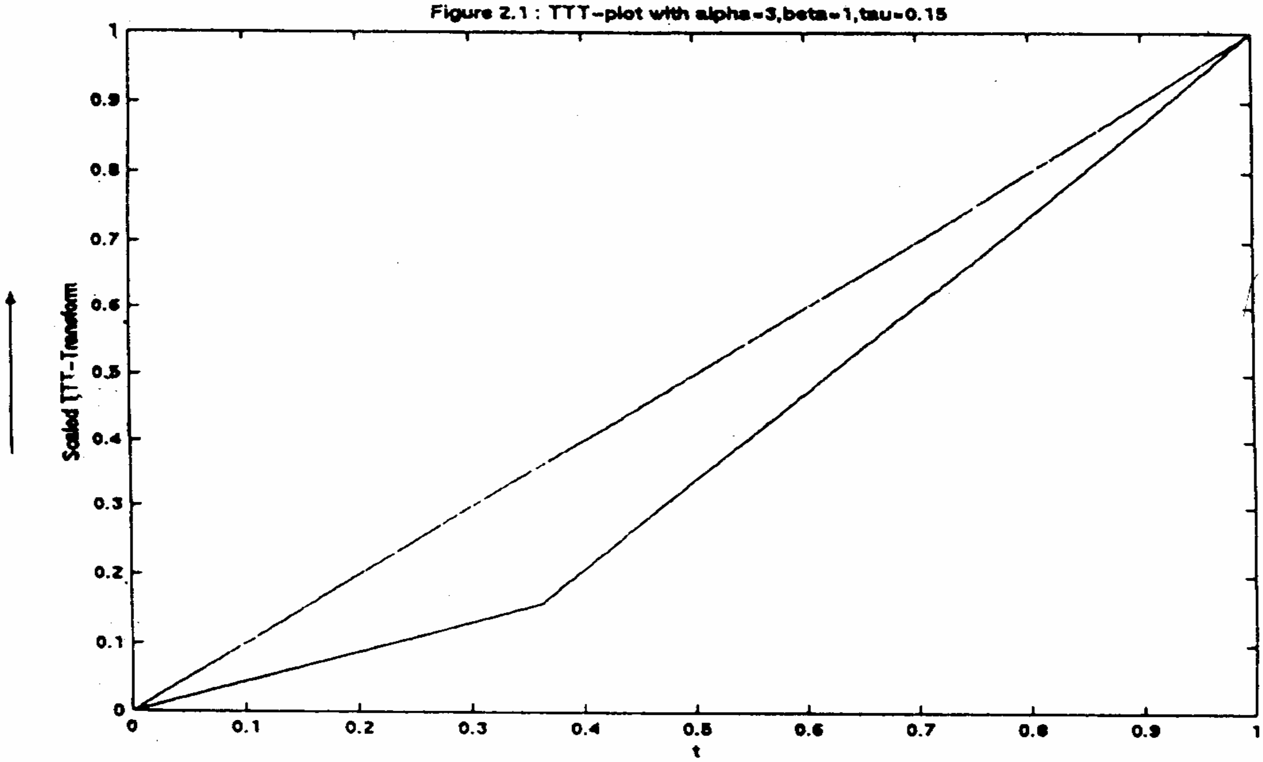
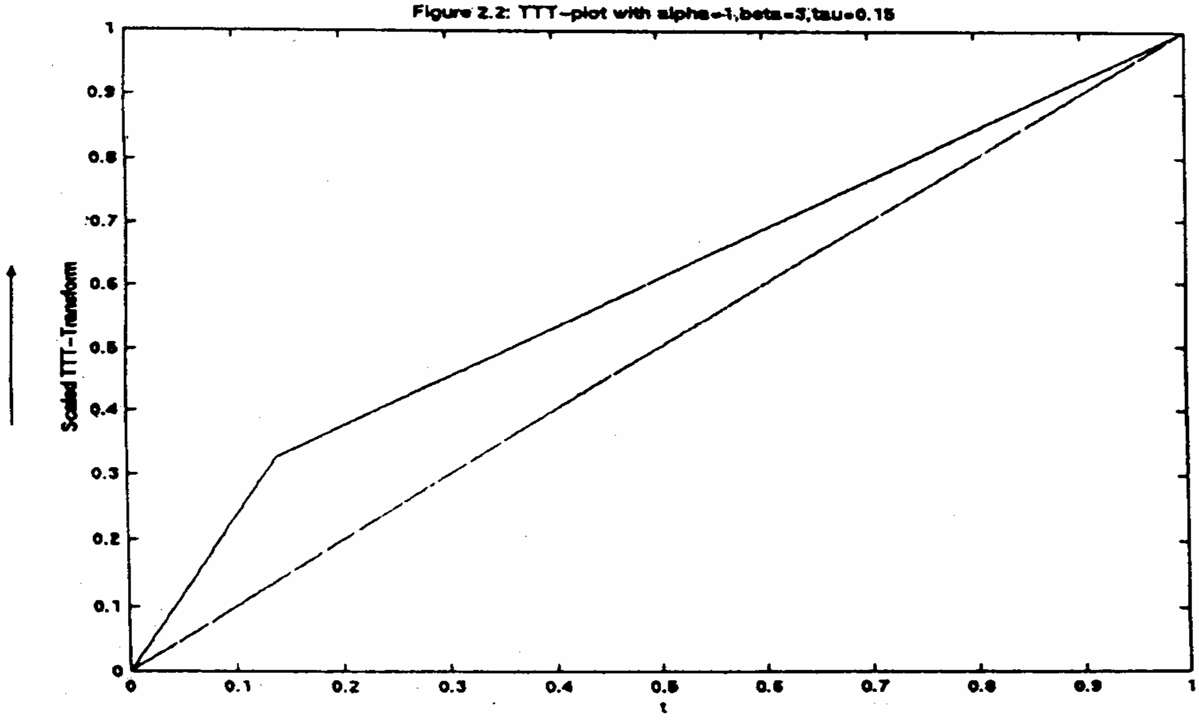


Figure 2.2: TTT-plot with alpha=1,beta=3,tau=0.15



2.2 Estimation of τ

Consider the estimation of τ under the assumption $\alpha > \beta$. It may be observed from the TTT - Transform that p_0 is the point of maximum of $t - \Psi_F(t)$. This motivates the following estimator of τ:

$$\hat{\tau} = F_n^{-1}(\hat{p}_0),$$

where $\hat{p}_0 = \frac{k}{n}$, and k is such that

$$2d_{k:n} - d_{k-1:n} - d_{k+1:n} = \max\{2 \leq i \leq n-1 : 2d_{i:n} - d_{i-1:n} - d_{i+1:n} > \epsilon_n\}$$

which is equivalent to

$$\hat{p}_0 = \inf\{0 < t < 1 : \Psi_n(t - 1/n) + \Psi_n(t + 1/n) - 2\Psi_n(t) > \epsilon_n\}$$

$d_{i:n} = i/n - u_i, i = 1, 2, \cdots, n$ and F_n is the empirical cdf of a sample of size n from model (1.1).

As in the case of other estimators of τ for this model (1.1) (see Ghosh, Joshi and Mukhopadhyay (1998)) this estimator also picked up local maximum far to the right of τ in simulated experiments. Hence, we restricted our estimation to models with $F(\tau) \leq 0.5$ (note that this assumption has also been made in Ghosh, Joshi and Mukhopadhyay (1998)). Under this assumption, we modify the estimator $\hat{\tau}$ as follows:

$$\hat{\tau}_{prop} = Min\{F_n^{-1}(\hat{p}_0), \quad F_n^{-1}(0.5)\}. \tag{2}$$

Remark 2.1 . The above estimator was derived under the assumption that $\alpha > \beta$. The estimator can be modified by replacing $d_{i:n}$ that is appearing in the estimator with $d_{i:n} = u_i - i/n, i = 1, \cdots, n$ when $\alpha < \beta$.

3 Consistency of $\hat{\tau}_{prop}$

In this section, we show that $\hat{\tau}_{prop}$ is strongly consistent for τ. The proof is based on the following result due to Langberg et. al. (1980)

$$Sup\{0 < t < 1 : \Psi_n(t) - \Psi_F(t)\} \to 0 \text{ a.s.} \tag{1}$$

Theorem 3.1. $\hat{\tau}_{prop} \to \tau$ a.s.

Proof. We first show that $\hat{p}_0 \to p_0$ a.s. Note that for sufficiently small $\epsilon > 0$ and for sufficiently large $n, p_0 + \epsilon - 1/n > p_0$, hence using (3.1), we have

$$\begin{aligned}&\Psi_n(p_0 + \epsilon - 1/n) + \Psi_n(p_0 + \epsilon + 1/n) - 2\Psi_n(p_0 + \epsilon)\\&= \Psi_F(p_0 + \epsilon - 1/n) + \Psi_F(p_0 + \epsilon + 1/n) - 2\Psi_F(p_0 + \epsilon) = 0.\end{aligned}$$

Thus, for large n, $\Psi_n(p_0+\epsilon-1/n)+\Psi_n(p_0+\epsilon+1/n)-2\Psi_n(p_0+\epsilon) \leq \epsilon_n$. It may be noted that for any $t > p_0+\epsilon$, $\Psi_n(t-1/n)+\Psi_n(t+1/n)-2\Psi_n(t) \leq \epsilon_n$ which implies that $\hat{p}_0 \leq p_0 + \epsilon$, and hence

$$P(\hat{p}_0 \leq p_0 + \epsilon) \to 1. \tag{2}$$

Now for suficiently small $\epsilon > 0$ and for sufficiently large $n, p_0 - \epsilon + 1/n < p_0$, hence using (3.1)

$$\begin{aligned}&\Psi_n(p_0 - \epsilon - 1/n) + \Psi_n(p_0 - \epsilon + 1/n) - 2\Psi_n(p_0 - \epsilon)\\&= \Psi_F(p_0 - \epsilon - 1/n) + \Psi_F(p_0 - \epsilon + 1/n) - 2\Psi_F(p_0 - \epsilon) = 0.\end{aligned}$$

Thus, for large n, $\Psi_n(p_0-\epsilon-1/n)+\Psi_n(p_0-\epsilon+1/n)-2\Psi_n(p_0-\epsilon) \leq \epsilon_n$. It may be noted that for any $t < p_0-\epsilon$, $\Psi_n(t-1/n)+\Psi_n(t+1/n)-2\Psi_n(t) \leq \epsilon_n$ which implies that $\hat{p}_0 \geq p_0 - \epsilon$, and hence

$$P(\hat{p}_0 \geq p_0 - \epsilon) \to 1. \tag{3}$$

The relation (3.2) and (3.3) prove that $\hat{p}_0 \to p_0$ a.s. Now the consistency of $\hat{\tau}_{prop}$ follows from the result that the inverse F_n^{-1} of the empirical cdf F_n converges uniformly and almost surely to F^{-1}. Hence the proof.

Remark 3.1. In the above theorem, strong consistency of the estimator was proved for the case $\alpha > \beta$. The proof in the case of $\alpha < \beta$ will follow in a similar manner.

4 Comparison of Estimators

In this section, we compare the proposed estimator $\hat{\tau}_{prop}$ with various estimators proposed by Basu, Ghosh and Joshi (1988) and Ghosh, Joshi and Mukhopadhyay (1993) and Ghosh, Joshi and Mukhopadhyay (1998). These estimators are given below:

$$\hat{\tau}_{BGJ1} = \inf\{t > 0 : y_n(t + h_n) - y_n(t) \leq h_n\lambda_0 + \epsilon_n\}$$
$$\hat{\tau}_{BGJ2} = \inf\{t > 0 : -y_n(t) - \log(1 - p_0) \leq \lambda_0(\hat{\xi}_{p_0} - t) + \epsilon_n\}$$

$$\hat{\tau}^{(s)}_{BGI2} = \begin{cases} \inf\{\tau \geq 0 : -y_n(t) + (\frac{1}{k})\Sigma y_n(t_{(i)}) \\ \qquad \leq \lambda_0((\frac{1}{k})\Sigma t_{(I)} - t) + \epsilon_n\} & \text{if } \leq \hat{\xi}_{p_0} \\ \hat{\xi}_{p_0} & \text{otherwise} \end{cases}$$

where $y_n(t) = -\log(\overline{F})_n(t), \overline{F}_n(t)$ being the empirical survival function, λ_0 is a least square estimate of the steady state hazard rate based on k order statistics $t([np_0] + 1), \cdots, t([np_1]), (p_0 < p_1 < 1)$ and the corresponding $y_n(\cdot)$ values, $\hat{\xi}_{p_0}$ is the p_0-th sample quantile, $h_n = n^{-1/4}, \epsilon_n = cn^{-1/2}\log n$, and the range of the summation is from $[np_0] + 1$ to $[np_1]$. Also, $\hat{\tau}_1$ and $\hat{\tau}_2$ represent the posterior mode and mean of τ based on the prior given below:

$$\pi(\alpha, \beta, \tau) = \frac{1}{\alpha\beta}, \quad 0 < a < \tau < b < \infty, \quad 0 < \beta_0 \leq \beta \leq \alpha < \infty.$$

Table 4.1 gives the mean and the MSE (in parenthesis) of these estimators for various values of the parameters (α, β, τ) computed across 1000 simulations of sample size $n = 100$ with $\epsilon_n = 0.05, h_n = n^{-1/4}, p_0 = 0.5, p_1 = 0.9, a = 0.05, \beta_0 = 0.05$. The upper bound b of τ is taken to be minimum of $\{\hat{\xi}_{p_0}, c_0/\beta_0\}$, where $c_0 = -\log(1 - p_0), p_0 = 0.05$. The proposed estimator $\hat{\tau}_{prop}$ is computed with $\epsilon_n = 0.05$.

From Table 4.1, it is clear that the proposed estimate $\hat{\tau}_{prop}$ has very less bias as compared to other estimators, however it has slightly higher (though comparable) MSE.

Table 4.2, below, gives the mean and the MSE (in paranthesis) of the estimator $\hat{\tau}_{prop}$ for various values (α, β, τ) with $\alpha < \beta$ computed across 1000 simulations of sample size $n = 100$ with $\epsilon_n = 0.05$.

TABLE 4.1 COMPARISON OF ESTIMATORS WITH $\alpha > \beta$

Mean and MSE of different estimates of τ

(α,β,τ)	$\hat{\tau}_{BGJ1}$	$\hat{\tau}_{BGJ2}$	$\hat{\tau}^{(s)}_{BGJ2}$	$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{\tau}_{prop}$
(3, 2 0, 0.10)	0.078140 (0.012753)	0.057305 (0.006746)	0.093246 (0.001961)	0.102316 (0.001961)	0.137340 (0.0002129)	0.1054 (0.0052)
(3, 2.0, 0.15)	0.1049[illegible]4 (0.015100)	0.077579 (0.009921)	0.111579 (0.011182)	0.124717 (0.002981)	0.146290 (0.000555)	0.1514 0.0073)
(3, 2.0, 0.20)	0.111652 (0.018660)	0.095320 (0.015337)	0.116434 (0.014677)	0.151055 (0.004419)	0.153734 (0.002565)	0.2075 (0.0053)
(3, 1.0, 0.10)	0.088946 (0.009513)	0.083324 (0.008579)	0.130174 (0.026171)	0.100355 (0.001143	0.122947 (0.001527)	0.1197 (0.0043)
(3, 1.0, 0.15)	0.123784 (0.005938)	0.108089 (0.005647)	0.141982 (0.014854)	0.145499 (0.001257)	0.157840 (0.000774)	0.1523 (0.0074)
(3, 1.0, 0.20)	0.172275 (0.006420)	0.148868 (0.004280)	0.170287 (0.006243)	0.184991 (0.001023)	0.188625 (0.000637)	0.21 (0.0077)
2, 1.0, 0.10	0.066162 (0.012013)	0.075726 (0.014784)	0.137726 (0.239181)	0.118568 (0.005721)	0.188821 (0.011579)	0.1056 (0.0044)
2, 1.0, 0.15	0.105740 (0.013299)	0.094265 (0.013921)	0.159268 (0.031041)	0.143342 (0.003090)	0.191047 (0.003816)	0.1403 (0.0091)
2, 1.0, 0.20	0.153471 (0.014977)	0.128155 (0.014681)	0.170236 (0.022642)	0.201996 (0.004524)	0.231774 (0.003046)	0.2233 (0.0159)
2, 0.5, 0.10	0.078410 (0.006312)	0.142974 (0.051890)	0.287965 (0.194533)	0.106296 (0.003670)	0.156138 (0.009588)	0.1060 (0.0037)
2, 0.5, 0.15	0.113983 (0.004772)	0.145358 (0.034068)	0.256493 (0.118936)	0.149228 (0.000833)	0.173840 (0.002145)	0.1529 (0.007)
2, 0.5, 0.20	0.169569 (0.009478)	0.165469 (0.018590)	0.256829 (0.067213)	0.191345 (0.000978)	0.209948 (0.001100)	0.2036 (0.0128)

Table 4.2 : Mean and MSE of different estimates of $\hat{\tau}_{prop}$ with $\alpha < \beta$.

Parameters	$\tau = 0.1$	$\tau = 0.15$	$\tau = 0.20$
$\alpha = 2, \beta = 3$	.0892 (.0069)	1486 (.0112)	.2054 (.0117)
$\alpha = 1, \beta = 3$	.1008 (.01)	.1464 (.0155)	.1856 (.0184)
$\alpha = 1, \beta = 2$	.1023 (.0116)	.1476 (.0208)	.1999 (.0215)
$\alpha = .5, \beta = 2$	.1053 (.01)	.1416 (.0209)	.2065 (.0335)

5 Concluding Remarks

All the available estimators are derived under the assumption that $\alpha > \beta$. The proposed estimator compares well for the case of $\alpha > \beta$, and can be extended to the case of $\alpha < \beta$. We have proved the strong consistency of the estimator and further asymptotic properties of the proposed estimator are being investigated.

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