A New Approach to System Reliability

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***Abstract*—Summary & Conclusions—Calculating system-reliability via the knowledge of structure function is not new. Such attempts have been made in the classical 1975 book by Barlow & Proschan. But they had to compromise with the increasing complexity of a system. This paper overcomes this problem through a new representation of the structure function, and demonstrates that the well-known systems considered in the state-of-art follow this new representation. With this new representation, the important reliability calculations, such as Birnbaum reliability-importance, become simple. The Chaudhuri, *et al.* (J. Applied Probability, 1991) bounds which exploit the knowledge of structure function were implemented by our simple and easy-to-use algorithm for some *s*-coherent structures,*viz*,*s*-series,*s*-parallel, 2-out-of-3:G, bridge structure, and a fire-detector system. The Chaudhuri bounds are superior to the Min–max and Barlow-Proschan bounds (1975).**

This representation is useful in implementing the Chaudhuri bounds, which are superior to the min–max, Barlow & Proschan bounds on the system reliability most commonly used in practice. With this representation of the structure function, the computation of important reliability measures such as the Birnbaum structural and reliability importance are easy.

The drawbacks of the Aven algorithm for computing system reliability are that it depends on the initial choice of some parameters, and can not deal with the case when the component survivor functions belong to the IFRA class of life distributions.

When the components have IFRA life, then the Chaudhuri bounds could be the best choice for the purpose of predicting reliability of a very complex *s*-coherent structure. The knowledge of some quantile of the component distributions is enough to obtain the Chaudhuri bounds whereas in order to implement by min–max bounds, a complete description of the component life distributions is required. The Barlow-Proschan bound is not valid for the important part of the system life, and is point-wise. The Chaudhuri bounds do fairly well for the useful part of the system life, and they coincide with the exact system reliability when the components are exponentially distributed. Thus, the use of Chaudhuri bounds is recommended for general use, especially when cost and/or time are critical.

The C-H-A algorithm (in this paper) is simple and easy to use. It depends on the knowledge of the path sets of a given structure. Standard software packages are available (CAFTAIN, Hoyland & Rausand, p 145, 1994) to provide the minimal path sets of any *s*-coherent system. The C-H-A algorithm has been programmed in SAS, S-PLUS, and MATLAB. Different computer codes of the algorithm are available on request from Prof. G. Chaudhuri. This method of predicting system reliability is under patent consideration at Indiana University, USA.

***Index Terms*—Birnbaum measure of reliability importance, increasing failure rate, structure function, system reliability.**

Manuscript received July 23, 1998; revised February 2, 2000 and December 12, 2000.

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Publisher Item Identifier S 0018-9529(01)06817-8.

Definitions:

Relevant Component: Component i is irrelevant to the structure Φ iff $\Phi(x)$ is constant in x_i; otherwise component i is relevant to the structure.

s-Coherent System: A system is s-coherent if all of its components are relevant, and if the structure function is increasing in each argument.

Path Set: A set of components of a system, which by functioning ensures that the system is functioning.

Minimal Path Set: A path set that cannot be reduced without losing its status as a path set.

Cut Set: A set of components, which by failing causes the system to fail.

Minimal Cut Set: A cut set that cannot be reduced without losing its status as a cut set.

Birnbaum Reliability-Importance: A measure of reliability importance of component i:

$$I_h(i) \equiv \frac{\partial h(\mathbf{p})}{\partial p_i} = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}).$$

Birnbaum Structural-Importance: A measure of structural importance of component i:

$$B_\Phi \equiv [h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})]_{p_j = \frac{1}{2}}, \quad j \neq i.$$

OR operation ($\oplus$): Performed with 2 binary numbers:

$$0 \oplus 0 = 0, 1 \oplus 0 = 0 \oplus 1 = 1 \oplus 1 = 1.$$

Acronyms[1] *:*

IFRA	increasing failure rate, average
OR	see OR operation in *Definitions*
B-P	Barlow and Proschan
C-H-A	Chaudhuri, Hu, and Afshar
MTTF	mean time to failure
Cdf	cumulative distribution function
Sf	survival function

Notation:

n	number of components
x	$(x_1, \ldots, x_n)$: states of the components
Φ(x)	system state
	1, if system is working
	0, otherwise
$X_(i)$	state of component i:
	1, if component i is working
	0, otherwise
p_i	$\Pr\{X_i = 1\}$: component i reliability
$\mathbf{p}$	$(p_1, \ldots, p_n)$
h	$\Pr\{\Phi(x)\}$: system reliability

[1]The singular & plural of an acronym are always spelled the same.

$(\cdot; x)'$	$(x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)'$
$h(\mathbf{p})$	implies: components are mutually *s*-independent
$I_h(i)$	Birnbaum reliability importance of component i
$B_\Phi(i)$	Birnbaum structural importance of component i
$F(t)$	system life Cdf
$\bar{F}(t)$	$1 - F(t)$
$r \times c$	order of a matrix: r, rows, c columns

I. Introduction

EXACT evaluation of system reliability is extremely difficult and sometimes impossible. Once one obtains the expression for the structure function, the system reliability computations become straightforward.

Attempts have been made to compute the exact system reliability of a complex system, for example, the algorithm [1] is based on minimal cut sets. The drawback this algorithm is that it depends on the initial choices of 2 parameters. The usual approach is to resort to bounds on system reliability [2].

This paper obtains a representation for the structure function of a *s*-coherent system, which is suitable for computer implementation.

Section II presents some definitions and known results.

Section III describes the main algorithm.

Section IV—

a) illustrates the algorithm through some well-known structures such as series, parallel, k-out-of-n:G, and a fire detector system,

b) computes some important reliability measures (Birnbaum's structural and reliability importance),

c) presents the Barlow & Proschan bound, the Chaudhuri bounds, and the min–max bounds; these bounds are implemented for the structures mentioned in a; the Chaudhuri bounds have an edge over the others.

II. Some Known Results

Let a *s*-coherent system consist of n *s*-independent components. If the life distributions of all these components are IFRA, then the system life distribution is also IFRA.

Ref [3] obtained bounds on the reliability of a *s*-coherent system consisting of *s*-independent components with IFRA distributions. The bounds are stated in theorem 1.

Theorem 1. Chaudhuri Bounds: Let $F_i(t)$ have IFRA life distributions and $0 < a < \infty$ for $i = 1, \ldots, n$, and let $h\left[\bar{F}_1(t), \ldots, \bar{F}_n(t)\right]$ denote the Sf of a *s*-coherent system.

$$h\left[\bar{F}_1(t), \ldots, \bar{F}_n(t)\right]$$
$$\begin{cases} \geq h(\left[\bar{F}_1(a)\right]^{t/a}, \ldots, \left[\bar{F}_n(a)\right]^{t/a}), & \text{for } t \leq a \\ \leq h(\left[\bar{F}_1(a)\right]^{t/a}, \ldots, \left[\bar{F}_n(a)\right]^{t/a}), & \text{for } t \geq a \end{cases}$$
$$\text{for } 0 < a < \infty \text{ and } t > 0$$

The elegance of this bound is that it is valid on the entire real line. The choice of a depends on the user's specification. This bound exploits the knowledge of some quantile of the component Cdf.

Theorem 2. Min–Max Bounds [2]: Let Φ be a *s*-coherent structure with state variables $x_1, \ldots, x_n$; let $P_1, \ldots, P_m$ denote the minimal path sets, let $K_1, \ldots, K_k$ denote the minimal cut sets.

$$\max_{1 \leq j \leq m} \left[\Pr\left\{\min_{i \in P_j}[X_i] = 1\right\}\right] \leq \Pr\{\Phi(\mathbf{X}) = 1\}$$
$$\leq \min_{1 \leq j \leq k} \left[\Pr\left\{\min_{i \in K_j}[X_i] = 1\right\}\right].$$

If, in addition, the $x_1, \ldots, x_n$ are associated, then

$$\max_{1 \leq j \leq m} \left[\prod_{i \in P_j} p_1\right] \leq \Pr\{\Phi(\mathbf{X}) = 1\} \leq \min_{1 \leq j \leq k} \left[\coprod_{i \leq j \leq k} p_1\right];$$
$$\coprod_{i=1}^{n} p_i \equiv 1 - (1 - p_1) \cdots (1 - p_n).$$

The *s*-independent r.v. are associated [2].

Theorem 3. Barlow-Proschan Bound [2]: Let F be IFRA with mean μ, and let $t > 0$ be fixed.

$$\bar{F} \leq \begin{cases} 1, & \text{for } t \leq \mu \\ \exp(-\omega \cdot t), & \text{for } t > \mu; \end{cases}$$
$$\omega > 0, \quad \exp(-\omega \cdot t) = 1 - \omega \cdot t.$$

III. The C-H-A Algorithm

Notation:

$\mathbf{V}$	vector of dimension n
v_i	element i of $\mathbf{V}$: 1, component i is in minimal path set 0, otherwise
$\mathbf{V}_j$	the $\mathbf{V}$ corresponding to minimal path set j, $j = 1, \ldots, m$
P	$(\mathbf{V}_1, \ldots, \mathbf{V} - m)$: $n \times m$ minimal path set matrix.

Step 1) Identify the minimal path-sets of the *s*-coherent structure under study. Generate P.

Step 2) Select the columns of the minimal path-set matrix P in pairs and perform an OR operation on their respective rows. There are $\binom{m}{2}$ such column combinations. At the end of each OR operation, the resulting column is appended to P, leading to the matrix:

$$(P, P_1)_{n \times (m + \binom{m}{2})}.$$

In this operation, the order in which columns are chosen is not important.

Step 3) Repeat step 2, except take 3 columns of P at a time and do an OR operation on their respective rows. At the end of this step, there will be $\binom{m}{3}$ new columns to be appended to (P, P_1) and yield

$$(P, P_1, P_2)_{n \times (m + \binom{m}{2} + \binom{m}{3})}.$$

Step 4) Repeat step 2 taking i, $i = 4, \ldots, m$ columns of P at a time. In the very last step, all m columns of P will be OR'ed with each other, resulting in the design matrix:

$$D = (P, P_1, P_2, \ldots, P_{m-1})_{n \times (m + \binom{m}{2} + \binom{m}{3} + \cdots + \binom{m}{m})}$$
$$= (P, P_1, P_2, \ldots, P_{m-1})_{n \times (2^m - 1)}.$$

Step 5) Construct a vector $\mathbf{1}$ of 1's of dimension $2^m - 1$ whose:

first m elements are 1's,
next $\binom{m}{2}$ entries have signs $(-1)^{2-1} = -1$,
next $\binom{m}{3}$ entries have signs $(-1)^{3-1} = +1$,
$\ddots$
last entry has sign $(-1)^{m-1}$.

In general, the signs are determined according to the rule $(-1)^{i-1}$, where i is the number of columns of P that are taken at a time to be OR'ed in a particular step.

Step 6) Obtain the structure function of the system:

$$\Phi(\mathbf{x}) = \sum_{j=1}^{2^m-1} \mathbf{1}(j) \cdot \prod_{i=1}^{n} x_i^{D(i,j)},$$
$$D(i,j) \equiv \text{element } (i,j) \text{ of } D,$$
$$\mathbf{1}(j) \equiv \text{element } j \text{ of } \mathbf{1}. \tag{3-1}$$

Step 7) Hence, the system reliability is:

$$h(\mathbf{p}) = \sum_{j=1}^{2^m-1} \mathbf{1}(j) \cdot \prod_{i=1}^{n} p_i^{D(i,j)}, \quad 0 < p_i < 1;$$
$$\mathbf{p} \equiv \text{vector of } p_i. \tag{3-2}$$

Since the minimal path sets uniquely determine a s-coherent structure, then (3-1) is unique.

IV. Illustrative Examples

This section illustrates C-H-A through the following *s*-coherent structures: series, parallel, 2-out-of-3, and bridge. For a practical application, a fire detector system is considered. The Birnbaum reliability-importance of these systems are calculated.

A. Series System

The series system (see Fig. 1) has 2 *s*-independent Weibull components, with Sf

$$\exp\left(-\frac{t^{\alpha_i}}{\beta_i}\right), \quad i = 1, 2.$$

The structure function of the system is:

$$\Phi(\mathbf{x}) = x_1 x_2. \tag{4-1}$$

The algorithm steps are:

Step 1. The system has 1 path set: 1,2. Hence

$$P = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{2\times 1}.$$

Step 2. There is 1 column in P; hence the OR operation is not used.

$$D = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{1} = (1).$$

Step 3 – 5 are not necessary because there is only 1 column in P.

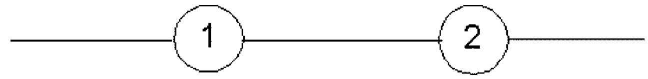

Fig. 1. Reliability block-diagram of the series structure.

Step 6.

$$\Phi(\mathbf{x}) = \sum_{j=1}^{1} 1(j) \cdot \prod_{i=1}^{2} x_i^{D(i,j)} = 1 \cdot x_1^1 \cdot x_2^1 = x_1 \cdot x_2;$$

which agrees with (4-1).

To compute the exact system reliability and its bounds, values for a, α, β are needed. The best candidate for a is the mean life of the system (MTTF):

$$a = \text{MTTF} = \int_0^\infty \bar{F}(t)dt = \int_0^\infty h[\bar{F}_1(t), \ldots, \bar{F}_n(t)]dt,$$

The values of α_i, β_i are given in the vectors for both components:

$$\alpha = [\,1.3\ 1.5\,]', \quad \beta = [\,1.0\ 1.0\,]'.$$

This integral can only be solved numerically, *eg*, by the trapezoidal or Simpson rules. The following steps 1 – 10 not only compute the MTTF, but they dynamically change the upper bound of the integral so that when the value of MTTF does not improve by more than a threshold, the integration stops.

Step 1) Set the lower & upper limits of the integral to $t_{\text{lb}} = 0$ and $t_{\text{ub}} = 1$, respectively. Also set the stepsize $= 0.25$, $old_{\text{MTTF}} = 0$, $\delta = 0.001$, $t = 0$.
Step 2) Set the time slice for integration to $\Delta t = (t_{\text{ub}} - 0)/100$.
Step 3) Compute $\bar{F}_i(t) = \exp(-t^{\alpha_i}/\beta_i)$,$i = 1, 2$ for both components.
Step 4) Use the $\bar{F}_i(t)$ values as $\mathbf{p}$ and compute $h(\mathbf{p})$ from(3-2).
Step 5) Save the current values of t and $h(\mathbf{p})$ in two arrays, x and h, respectively.
Step 6) Reset $t = t + \Delta t$.
Step 7) If $t \le t_{\text{ub}}$, then go to step 3; otherwise, go to step 8.
Step 8) Numerically integrate to compute MTTF using the x, h arrays; see step 5.
Step 9) If $|\text{MTTF} - \text{old\_MTTF}| < \delta$, then stop; otherwise, go to step 10.
Step 10) Set $\text{old\_MTTF} = \text{MTTF}$. Set the new $t_{\text{ub}} = t_{\text{ub}} +$ stepsize; then go to step 2.

Once the $a = \text{MTTF}$ is computed, the h array contains the exact reliability function over the time interval from 0 to the last value of t_{ub}.

To compute the reliability bounds, use the following steps (slightly modified from the previous 10 steps).

Step 1) Set $t = 0$.
Step 2) Compute the values of

$$[\bar{F}_i(a)]^{t/a} = \left[\exp\left(-\frac{t^{\alpha_i}}{\beta_i}\right)\right]^{t/a}, \quad i = 1, 2.$$

Step 3) Use the $[\bar{F}_i(a)]^{t/a}$ values as $\mathbf{p}$, and compute $h(\mathbf{p})$ as in step 4.

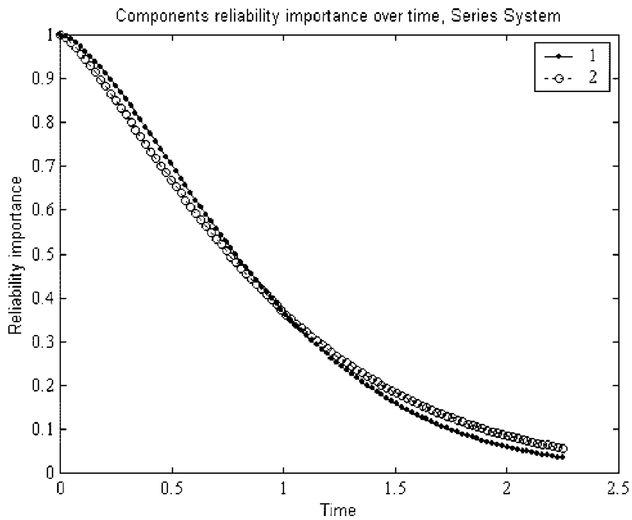

Fig. 2. Components 1 & 2 reliability-importance *vs* time. The slight difference between the 2 reliability-importance functions is due to the different values of α.

Step 4) Save $h(\mathbf{p})$ in array b.

Step 5) Reset $t = t + \Delta t$; Δt is the same as that of the latest iteration of MTTF computation.

Step 6) If $t \leq t_{\text{ub}}$, go to step 2; otherwise, stop.

The definitions of the variables are:

pathset: minimal path-set matrix, P
cutset: minimal cut-set matrix
D: design matrix, D
reliab: system reliability
alpha: shape parameter of the Weibull distribution
beta: scale parameter of the Weibull distribution
last_t: t_{ub} in the reliability calculation algorithm, the largest value of t at which the area under the exact reliability curve changes less than a very small amount.

The information from the computer printout for Matlab implementation of the algorithm is:

$$\begin{aligned}
&\text{pathset} = \\
&\quad 1 \\
&\quad 1 \\
&\text{outset} = \\
&\quad 1 \quad 0 \\
&\quad 0 \quad 1 \\
&D = \\
&\quad 1 \\
&\quad 1 \\
&\text{alpha} = [1.3 \quad 1.5]' \\
&\text{beta} = [1 \quad 1]' \\
&\text{last_t} = 2.25
\end{aligned}$$

Fig. 2 plots the component-reliability importance as a function of time. Table I lists the values of exact system reliability and its bounds at several time points.

Fig. 3 compares the exact reliability function, Min–max bounds, B-P bound, and Chaudhuri's bounds as a function of time. It shows that, for a series structure, the Max bound is the same as the exact reliability.

TABLE I
COMPARISON OF THE EXACT SYSTEM RELIABILITY, CHAUDHURI, B-P, AND MIN-MAX BOUNDS

Time	Exact	Bounds			
t	reliability	Min	Max	Chaudhuri	B-P
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1125	0.9083	0.9083	0.9433	0.8370	1.0000
0.2250	0.7784	0.7784	0.8660	0.7005	1.0000
0.3375	0.6442	0.6442	0.7838	0.5863	1.0000
0.4500	0.5189	0.5189	0.7018	0.4907	1.0000
0.5625	0.4085	0.4085	0.6229	0.4107	0.9348
0.6750	0.3152	0.3152	0.5489	0.3438	0.6643
0.7875	0.2389	0.2389	0.4804	0.2877	0.4724
0.9000	0.1780	0.1780	0.4181	0.2408	0.3451
1.0125	0.1307	0.1307	0.3610	0.2015	0.2574
1.1250	0.0945	0.0945	0.3032	0.1687	0.1950
1.2375	0.0675	0.0675	0.2524	0.1412	0.1497
1.3500	0.0476	0.0476	0.2083	0.1182	0.1160
1.4625	0.0331	0.0331	0.1706	0.0989	0.0907
1.5750	0.0228	0.0228	0.1385	0.0828	0.0714
1.6875	0.0155	0.0155	0.1117	0.0693	0.0565
1.8000	0.0104	0.0104	0.0894	0.0580	0.0449
1.9125	0.0070	0.0070	0.0710	0.0485	0.0359
2.0250	0.0046	0.0046	0.0560	0.0406	0.0287
2.1375	0.0030	0.0030	0.0439	0.0340	0.0231
2.2500	0.0019	0.0019	0.0342	0.0285	0.0186

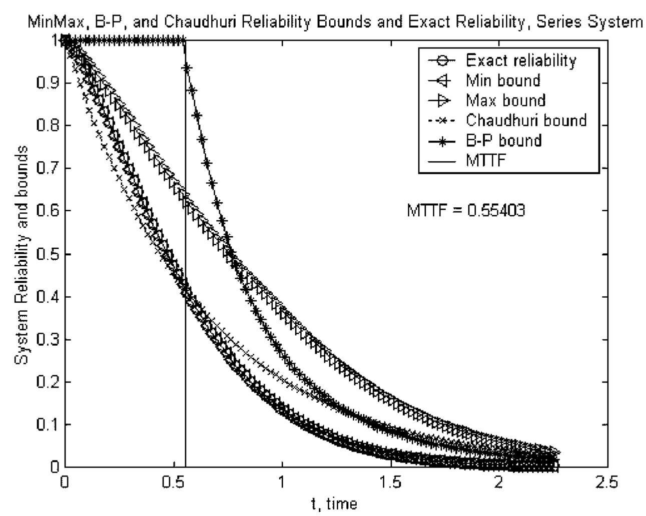

Fig. 3. System reliability and various bounds *vs* time.

B. Parallel System

Fig. 4 shows a parallel structure with 2 *s*-independent Weibull components; the component Sf are

$$\exp\left(-\frac{t^{\alpha_i}}{\beta_i}\right), \quad i = 1, 2.$$

The system structure-function is

$$\Phi(\mathbf{x}) = x_1 + x_2 - x_1 \cdot x_2.$$

The system's minimal path-sets are: $\{1\}$, $\{2\}$.

Step 1. The P matrix is:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}.$$

Step 2:

$$D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

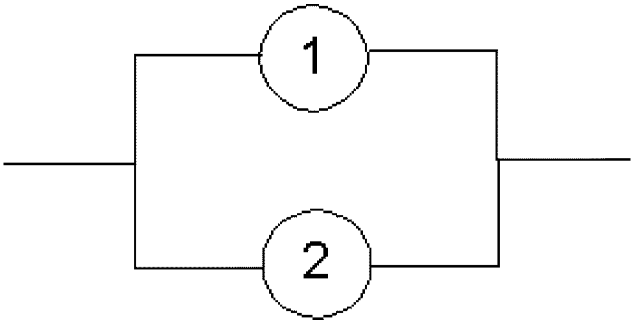

Fig. 4. Reliability block diagram of a parallel structure.

Step 3: Because P has only 2 columns, the D is given in step 2, and step 4 is not necessary.
Step 5: $\mathbf{1} = [1 \quad 1 - 1]'$.
Step 6:

$$\begin{aligned}\Phi(\mathbf{x}) = &\sum_{j=1}^{3} \mathbf{1}(j) \prod_{i=1}^{2} x_i^{D(i,j)} = \mathbf{1}(1) \cdot x_1^{D(1,1)} \cdot x_2^{D(2,1)} \\ &+ \mathbf{1}(2) \cdot x_1^{D(1,2)} \cdot x_2^{D(2,2)} + \mathbf{1}(3) \cdot x_1^{D(1,3)} \cdot x_2^{D(2,3)} \\ =&1 \cdot x_1^1 \cdot x_2^0 + 1 \cdot x_1^0 \cdot x_2^1 + (-1) \cdot x_1^1 \cdot x_2^1 \\ =&x_1 + x_2 - x_1 \cdot x_2.\end{aligned}$$

which agrees with (4–2).

Similarly, the Birnbaum reliability importance is computed as in the series system, Fig. 5. Table II lists the values of the exact reliability and its various bounds at several time points. The results of these computations are:

$$\begin{aligned}&\text{pathset} = \\ &\quad 1 \quad 0 \\ &\quad 0 \quad 1 \\ &\text{outset} = \\ &\quad 1 \\ &\quad 1 \\ &D = \\ &\quad 1 \quad 0 \quad 1 \\ &\quad 0 \quad 1 \quad 1 \\ &\text{alpha} = [1.3 \quad 1.5]' \\ &\text{beta} = [1 \quad 1]' \\ &\text{last\_t} = 4\end{aligned}$$

Fig. 6 compares various bounds with respect to the exact reliability. For the parallel structure, the Min bound is the same as the exact reliability.

C. 2-out-of-3:G System

Consider a 2-out-of-3:G structure with 3 *s*-independent Weibull components, Fig. 7, with Sf $\exp(-t^{\alpha_i}/\beta_i)$, $i = 1, 2, 3$. The system structure-function is:

$$\Phi(\mathbf{x}) = x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 - 2x_1 \cdot x_2 \cdot x_3. \quad (4\text{-}3)$$

The system has minimal path-sets: $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$.

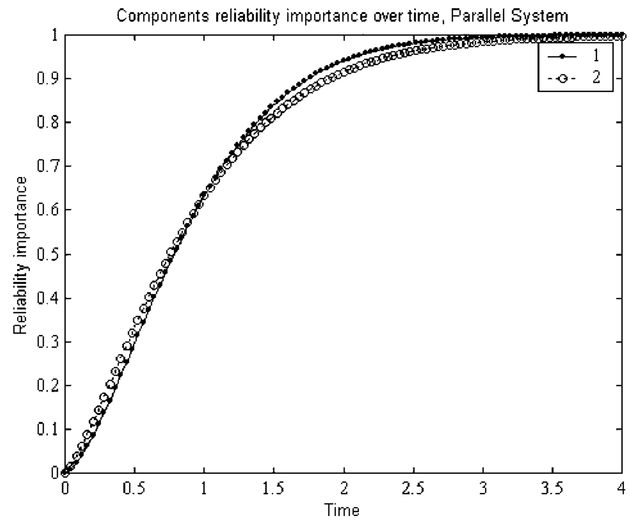

Fig. 5. Components 1 & 2 reliability importance *vs* time.

TABLE II
COMPARISON OF THE EXACT SYSTEM RELIABILITY, CHAUDHURI, B-P, AND MIN-MAX BOUNDS

Time t	Exact reliability	Bounds Min	Max	Chaudhuri	B-P
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2000	0.9901	0.9144	0.9901	0.9610	1.0000
0.4000	0.9414	0.7765	0.9414	0.8732	1.0000
0.6000	0.8504	0.6283	0.8504	0.7664	1.0000
0.8000	0.7308	0.4889	0.7308	0.6574	1.0000
1.0000	0.6004	0.3679	0.6004	0.5548	1.0000
1.2000	0.4745	0.2815	0.4745	0.4628	1.0000
1.4000	0.3628	0.2125	0.3628	0.3827	0.8161
1.6000	0.2697	0.1585	0.2697	0.3144	0.6186
1.8000	0.1958	0.1168	0.1958	0.2570	0.4760
2.0000	0.1393	0.0852	0.1393	0.2093	0.3723
2.2000	0.0975	0.0616	0.0975	0.1699	0.2950
2.4000	0.0673	0.0441	0.0673	0.1376	0.2363
2.6000	0.0460	0.0313	0.0460	0.1113	0.1910
2.8000	0.0311	0.0221	0.0311	0.0898	0.1555
3.0000	0.0209	0.0154	0.0209	0.0725	0.1274
3.2000	0.0139	0.0107	0.0139	0.0584	0.1050
3.4000	0.0093	0.0074	0.0093	0.0470	0.0869
3.6000	0.0061	0.0051	0.0061	0.0378	0.0722
3.8000	0.0040	0.0034	0.0040	0.0304	0.0602
4.0000	0.0027	0.0023	0.0027	0.0245	0.0503

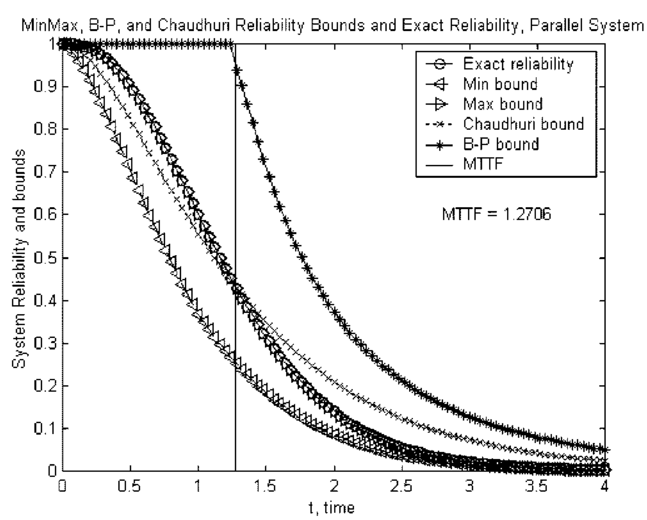

Fig. 6. System reliability and various bounds *vs* time.

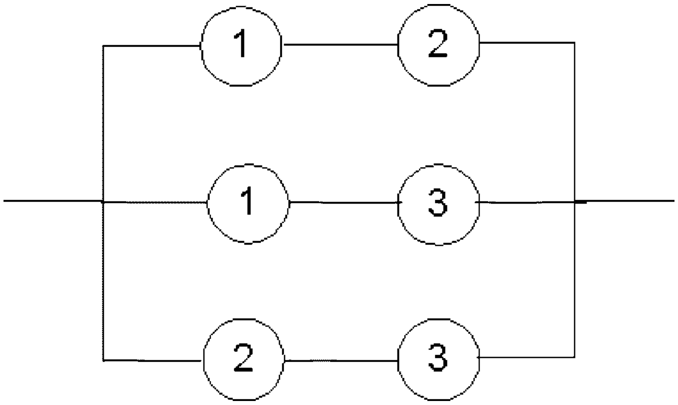

Fig. 7. Reliability block diagram of a 2-out-of-3 structure.

Step 1. The P matrix is:

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Steps 2 – 4: The final D matrix is:

$$D = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{3\times 7}$$

Step 5: $\mathbf{1} = [\ 1\ 1\ 1 -1 -1 -1\ 1\ ]'$.

Step 6: The structure function is:

$$\begin{aligned}\Phi(\mathbf{x}) =& 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^0 + 1 \cdot x_1^1 \cdot x_2^0 \cdot x_3^1 + 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^0 \\ &+ 1 \cdot x_1^0 \cdot x_2^1 \cdot x_3^1 - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \\ &- 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \\ =& x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 - 2x_1 \cdot x_2 \cdot x_3,\end{aligned}$$

which agrees with (4.3).

Fig. 8 shows the Birnbaum reliability-importance. Table III lists the values of the exact reliability and its various bounds at several time points. The results of these computations are:

$$\begin{aligned}&\text{pathset} = \\ &\quad 1\quad 1\quad 0 \\ &\quad 1\quad 0\quad 1 \\ &\quad 0\quad 1\quad 1 \\ &\text{outset} = \\ &\quad 1\quad 1\quad 0 \\ &\quad 1\quad 0\quad 1 \\ &\quad 0\quad 1\quad 1 \\ &D = \\ &\quad 1\quad 1\quad 0\quad 1\quad 1\quad 1\quad 1 \\ &\quad 1\quad 0\quad 1\quad 1\quad 1\quad 1\quad 1 \\ &\quad 0\quad 1\quad 1\quad 1\quad 1\quad 1\quad 1 \\ &\text{alpha} = [1.3\quad 1.5\quad 1.7]' \\ &\text{beta} = [1\quad 1\quad 1]' \\ &\text{last\_t} = 2.5\end{aligned}$$

Fig. 9 compares various bounds with respect to the exact reliability.

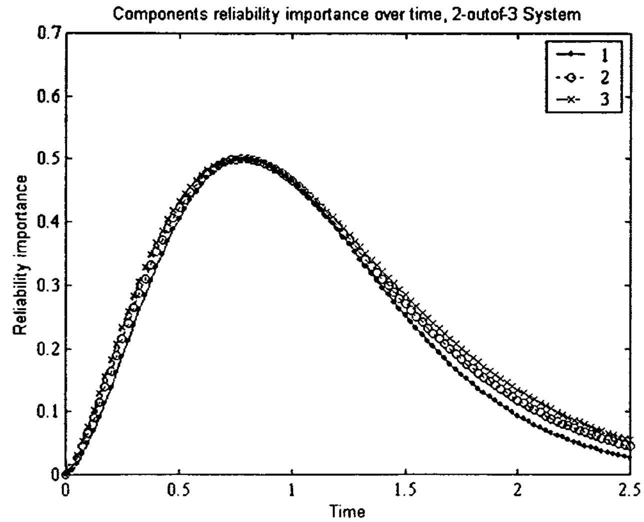

Fig. 8. Components 1 – 3 reliability-importance *vs* time.

TABLE III
COMPARISON OF THE EXACT SYSTEM RELIABILITY, CHAUDHURI, B-P, AND MIN-MAX BOUNDS

Time	Exact	Bounds			
t	reliability	Min	Max	Chaudhuri	B-P
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1250	0.9943	0.9293	0.9972	0.9675	1.0000
0.2500	0.9610	0.8027	0.9821	0.8916	1.0000
0.3750	0.8900	0.6581	0.9500	0.7958	1.0000
0.5000	0.7859	0.5162	0.9006	0.6945	1.0000
0.6250	0.6620	0.3891	0.8367	0.5960	1.0000
0.7500	0.5332	0.2829	0.7624	0.5050	1.0000
0.8750	0.4120	0.1988	0.6822	0.4237	0.8994
1.0000	0.3064	0.1353	0.6004	0.3526	0.6936
1.1250	0.2201	0.0945	0.5086	0.2915	0.5367
1.2500	0.1531	0.0650	0.4218	0.2397	0.4221
1.3750	0.1034	0.0439	0.3430	0.1963	0.3364
1.5000	0.0681	0.0293	0.2739	0.1601	0.2710
1.6250	0.0437	0.0192	0.2152	0.1301	0.2203
1.7500	0.0275	0.0125	0.1664	0.1055	0.1804
1.8750	0.0169	0.0080	0.1270	0.0854	0.1487
2.0000	0.0102	0.0050	0.0956	0.0689	0.1232
2.1250	0.0061	0.0031	0.0712	0.0555	0.1026
2.2500	0.0036	0.0019	0.0525	0.0447	0.0858
2.3750	0.0021	0.0012	0.0383	0.0359	0.0720
2.5000	0.0012	0.0007	0.0277	0.0288	0.0606

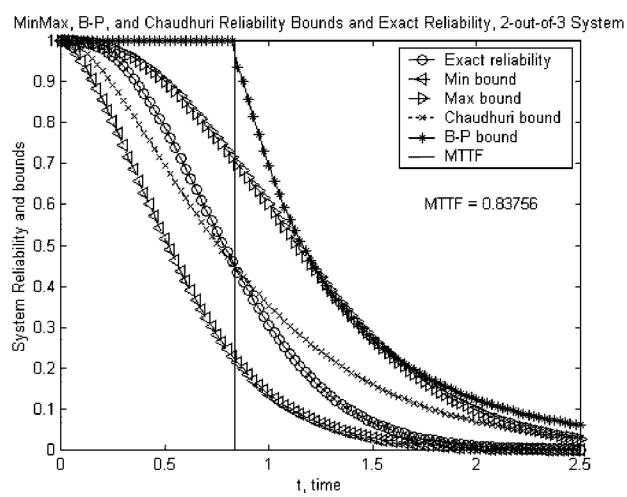

Fig. 9. System reliability and various bounds *vs* time.

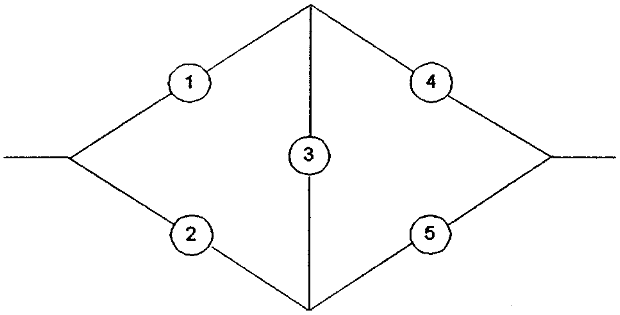

Fig. 10. Reliability block diagram of the bridge structure.

D. Bridge System

The bridge structure has 5 *s*-independent Weibull components (see Fig. 10) with $\text{Sf} = \exp(-t^{\alpha_i}/\beta_i), i = 1, \ldots, 5$. The structure function is:

$$\begin{aligned}\Phi(x) = & x_1 \cdot x_4 + x_2 \cdot x_5 + x_1 \cdot x_3 \cdot x_5 + x_2 \cdot x_3 \cdot x_4 \\ & - x_1 \cdot x_2 \cdot x_4 \cdot x_5 - x_1 \cdot x_3 \cdot x_4 \cdot x_5 \\ & - x_1 \cdot x_2 \cdot x_3 \cdot x_4 - x_1 \cdot x_2 \cdot x_3 \cdot x_5 \\ & + 2x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5. \end{aligned} \tag{4-4}$$

The system has the minimal path sets: $\{1,4\}$, $\{2,5\}$, $\{1,3,5\}$, $\{2,3,4\}$.

Step 1: The P matrix is:

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}_{5\times 4}.$$

Step 2 – 4. See equation at the bottom of the page.

Step 5: $\mathbf{1} = [1\ 1\ 1\ 1 -1 -1 -1 -1 -1 -1 -1\ 1\ 1\ 1\ 1 -1]'$.

Step 6:

$$\begin{aligned}\Phi(\mathbf{x}) = & 1 \cdot x_1^1 \cdot x_2^0 \cdot x_3^0 \cdot x_4^1 \cdot x_5^0 + 1 \cdot x_1^0 \cdot x_2^1 \cdot x_3^0 \cdot x_4^0 \cdot x_5^1 \\ & + 1 \cdot x_1^1 \cdot x_2^0 \cdot x_3^1 \cdot x_4^0 \cdot x_5^1 + 1 \cdot x_1^0 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^0 \\ & - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^0 \cdot x_4^1 \cdot x_5^1 - 1 \cdot x_1^1 \cdot x_2^0 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 \\ & - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^0 - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^0 \cdot x_5^1 \\ & - 1 \cdot x_1^0 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 \\ & + 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 + 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 \\ & + 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 + 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 \\ & - 1 \cdot x_1^1 \cdot x_2^1 \cdot x_3^1 \cdot x_4^1 \cdot x_5^1 \\ = & x_1 \cdot x_4 + x_2 \cdot x_5 + x_1 \cdot x_3 \cdot x_5 + x_2 \cdot x_3 \cdot x_4 \\ & - x_1 \cdot x_2 \cdot x_4 \cdot x_5 - x_1 \cdot x_3 \cdot x_4 \cdot x_5 \\ & - x_1 \cdot x_2 \cdot x_3 \cdot x_4 - x_1 \cdot x_2 \cdot x_3 \cdot x_5 \\ & - x_2 \cdot x_3 \cdot x_4 \cdot x_5 + 2x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \end{aligned}$$

which agrees with (4-4).

The results of these computations are:

$$\text{phathset} = \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

$$\text{cutset} = \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

$$D = \text{ colums } 1-12$$

$$\begin{matrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$$\text{colums } 13-15$$

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

$$\text{alpha} = [1.3\ 1.5\ 1.7\ 2.1\ 2.3]'$$

$$\text{beta} = [1\ 1\ 1\ 1\ 1]'$$

$$\text{last\_t} = 2.25$$

Fig. 11 shows the Birnbaum reliability-importance.

Table IV lists the exact reliability and its bounds at several time points.

Fig. 12 compares various bounds with respect to the exact reliability.

E. Fire-Detector System

This pneumatic system is from [4]; it consists of 3 parts: heat detection, smoke detection, and a manually-operated alarm button. Fig. 13 is the system-reliability block diagram.

In the heat-detection section, there is a circuit with 4 *s*-identical fuse plugs, FP1, FP2, FP3, FP4, which forces the air out of the circuit if they experience temperatures more than 72°C. The circuit is connected to a pressure switch (PS). The PS begins functioning when at least 1 of the plugs begins working, and

$$D = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

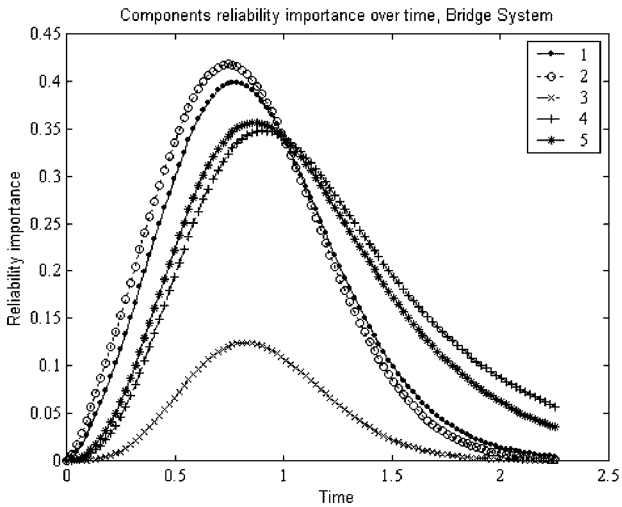

Fig. 11. Components 1 – 5 reliability-importance *vs* time.

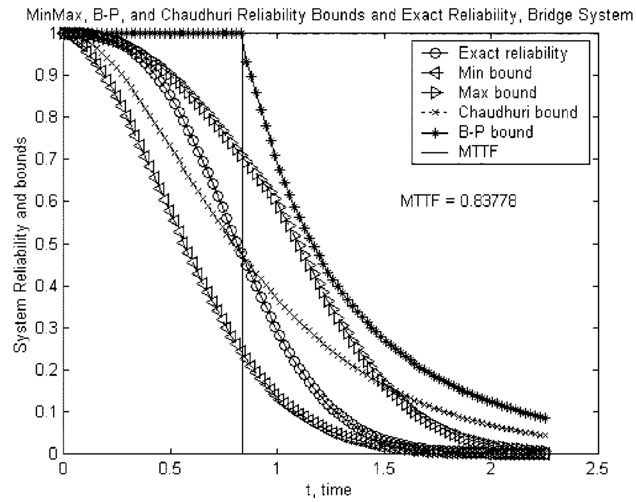

Fig. 12. System reliability and various bounds *vs* time.

TABLE IV
COMPARISON OF THE EXACT SYSTEM RELIABILITY, CHAUDHURI, B-P, AND MIN-MAX BOUNDS

Time	Exact	Bounds			
t	reliability	Min	Max	Chaudhuri	B-P
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1125	0.9978	0.9567	0.9979	0.9813	1.0000
0.2250	0.9845	0.8702	0.9864	0.9299	1.0000
0.3375	0.9505	0.7571	0.9615	0.8559	1.0000
0.4500	0.8870	0.6305	0.9223	0.7693	1.0000
0.5625	0.7910	0.5025	0.8702	0.6784	1.0000
0.6750	0.6680	0.3831	0.8080	0.5893	1.0000
0.7875	0.5316	0.2791	0.7388	0.5055	1.0000
0.9000	0.3978	0.1942	0.6659	0.4294	0.8590
1.0125	0.2802	0.1297	0.5876	0.3619	0.6759
1.1250	0.1862	0.0866	0.4725	0.3030	0.5370
1.2375	0.1173	0.0559	0.3638	0.2524	0.4324
1.3500	0.0702	0.0349	0.2682	0.2094	0.3519
1.4625	0.0402	0.0210	0.1895	0.1731	0.2891
1.5750	0.0220	0.0123	0.1285	0.1427	0.2393
1.6875	0.0116	0.0069	0.0837	0.1174	0.1993
1.8000	0.0059	0.0038	0.0525	0.0965	0.1670
1.9125	0.0029	0.0020	0.0317	0.0792	0.1406
2.0250	0.0014	0.0010	0.0185	0.0649	0.1189
2.1375	0.0006	0.0005	0.0104	0.0532	0.1009
2.2500	0.0003	0.0002	0.0057	0.0435	0.0859

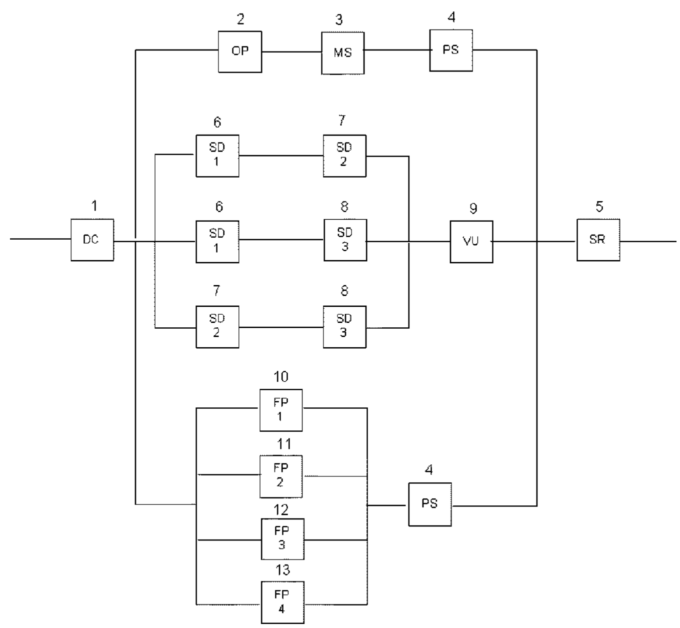

Fig. 13. Reliability block-diagram of the fire-detector structure.

transmits a signal to the start relay (SR) to produce an alarm and thereby causing a system shut-down.

The smoke-detection section has 3 smoke detectors, SD1, SD2, SD3, which are connected to a voting unit (VU) through a logical 2-out-of-3:G system. Thus at least 2 smoke detectors must give a fire signal before the fire alarm is activated.

For the successful transmission of an electric signal from heat-detector and/or smoke-detector, the DC source must be working.

In the manual activation section, there is an operator OP, who should always be present. If the operator observes a fire, then the operator turns-on the manual-switch (MS) to relieve pressure in the circuit of the heat-detection section. This activates the PS switch, which in turn gives an electric signal to SR. Of course, DC must be functioning.

The system has 8 minimal path-sets: $\{1,2,3,4,5\}$, $\{1,5,6,7,9\}$, $\{1,5,6,8,9\}$, $\{1,5,7,8,9\}$, $\{1,4,5,10\}$, $\{1,4,5,11\}$, $\{1,4,5,12\}$, $\{1,4,5,13\}$.

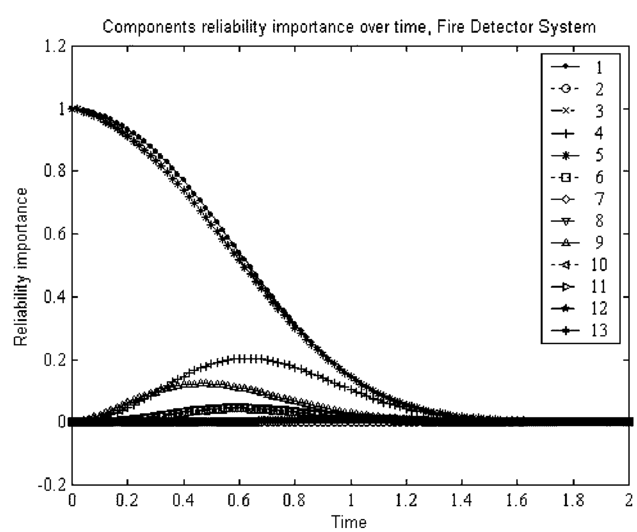

Fig. 14. Components 1 – 13 reliability importance *vs* time.

TABLE V
COMPARISON OF THE EXACT SYSTEM RELIABILITY, CHAUDHURI, B-P, AND MIN-MAX BOUNDS

Time	Exact	Bounds			
t	reliability	Min	Max	Chaudhuri	B-P
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1000	0.9494	0.9215	0.9689	0.8723	1.0000
0.2000	0.8536	0.7748	0.9144	0.7543	1.0000
0.3000	0.7327	0.6054	0.8485	0.6464	1.0000
0.4000	0.5989	0.4422	0.7765	0.5491	1.0000
0.5000	0.4640	0.3029	0.7022	0.4628	1.0000
0.6000	0.3398	0.1950	0.6283	0.3872	0.6936
0.7000	0.2347	0.1181	0.5567	0.3219	0.4946
0.8000	0.1526	0.0673	0.4889	0.2662	0.3624
0.9000	0.0932	0.0362	0.4258	0.2189	0.2710
1.0000	0.0533	0.0183	0.3679	0.1793	0.2060
1.1000	0.0285	0.0091	0.3085	0.1462	0.1585
1.2000	0.0142	0.0043	0.2558	0.1189	0.1232
1.3000	0.0066	0.0019	0.2097	0.0963	0.0966
1.4000	0.0029	0.0008	0.1700	0.0778	0.0763
1.5000	0.0012	0.0003	0.1364	0.0627	0.0606
1.6000	0.0005	0.0001	0.1082	0.0504	0.0483
1.7000	0.0002	0.0000	0.0850	0.0404	0.0387
1.8000	0.0001	0.0000	0.0661	0.0324	0.0311
1.9000	0.0000	0.0000	0.0509	0.0259	0.0251
2.0000	0.0000	0.0000	0.0388	0.0207	0.0203

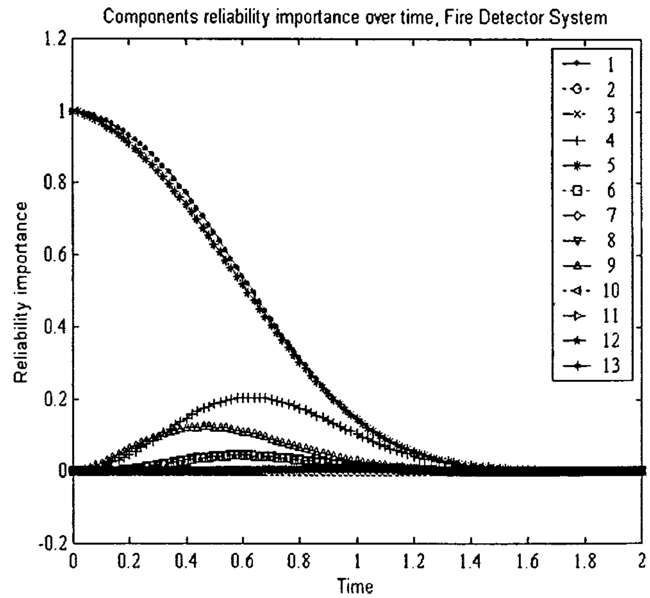

Fig. 15. System-reliability and various bounds *vs* time.

Since computing this system is involved & lengthy, only a partial printout is provided here. For example, the D matrix for this system has $2^8 - 1 = 255$ columns. Fig. 14 shows the Birnbaum reliability-importance. Table V lists the values of the exact reliability and its bounds at several time points. The computation-results are:

pathset =

1 1 1 1 1 1 1 1
1 0 0 0 0 0 0 0
1 0 0 0 0 0 0 0
1 0 0 0 1 1 1 1
1 1 1 1 1 1 1 1
0 1 1 0 0 0 0 0
0 1 0 1 0 0 0 0
0 0 1 1 0 0 0 0
0 1 1 1 0 0 0 0
0 0 0 0 1 0 0 0
0 0 0 0 0 1 0 0
0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 1

cutset =

colums1 − 12

1 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 0 0 0
0 0 0 0 0 0 0 0 1 1 1 1
0 0 1 1 1 1 0 0 0 0 0 0
0 1 0 0 0 0 0 0 0 0 0 0
0 0 1 0 1 0 1 1 0 1 1 0
0 0 1 1 0 0 1 0 1 1 0 1
0 0 0 1 1 0 0 1 1 0 1 1
0 0 0 0 0 1 0 0 0 0 0 0
0 0 0 0 0 0 1 1 1 1 1 1
0 0 0 0 0 0 1 1 1 1 1 1
0 0 0 0 0 0 1 1 1 1 1 1
0 0 0 0 0 0 1 1 1 1 1 1

colums 13 − 14

0 0
1 0
0 1
0 0
0 0
0 0
0 0
0 0
1 1
1 1
1 1
1 1

$\text{alpha} = [1.5\ 1.5\ 1.6\ 1.6\ 1.7\ 1.7\ 1.8\ 1.8\ 1.9\ 2.0\ 2.1\ 2.2\ 2.3]'$

$\text{beta} = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]'$

$\text{last\_t} = 2$

Fig. 15 compares various bounds with respect to the exact reliability.

ACKNOWLEDGMENT

This paper is based on a project given by Professor Gopal Chaudhuri to his students (Kuolung Hu and Dr. Nader Afshar) while teaching a graduate course on Reliability Theory, 1998 Spring, at Indiana University—Purdue University, in Indianapolis. We are pleased to thank Dr. Jim Maxwell for some stimulating discussions. We are grateful to the editors, and the managing editor in particular, for excellent comments & suggestions that immensely improved the presentation of this paper.

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