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Bayes Estimation with Bivariate Prior in $M/M/1$ Queues

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Abstract

Bayes estimators of different queueing performance measures are derived in steady state by recording system size from each of n iid $M/M/1$ queues. The Bayes estimators are obtained under both squared error loss function and precautionary loss function with a bivariate distribution, Beta-Stacy as prior with natural restriction $0 < \lambda < \mu$ where λ and μ are arrival rate and service rate respectively. A comprehensive simulation results are also shown at the last section.

Keywords and Phrases: Bayes estimator; $M/M/1$ queue; Queue length; Beta-Stacy distribution; Monte Carlo simulation.
AMS 2010 Subject Classifications: Primary 60K25; Secondary 90B22.

1 Introduction

Main purpose of queueing theory is to develop models to predict the behaviour of systems that attempt to provide service for randomly arriving demands. Queues (or waiting lines) help facilities or systems to provide service in an orderly fashion. Any conclusion about a waiting line problem comes from analyzing the model representing the queue. The analysis is based on building a mathematical model representing a process of arrival of customers who join the queue, the rules by which they are allowed into service and the time it takes to serve the customers. Queueing theory embodies the full gamut of such models covering all perceivable systems which incorporate characteristics of a queue.

Plethora of practical applications, such as manufacturing and production systems, communication and networking systems, transportation systems, healthcare systems and facility design (banks, post offices, amusement parks, restaurants etc.) are often modeled as queueing systems to investigate their operational performance such as queue lengths and waiting times. Many performance measures of queueing systems are important indicators of their productivity and also the critical dimensions of the service quality. These performance measures are often

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quantitatively estimated using corresponding queueing performance metrics (QPMs), such as average queue length (L_q), average system length (L_s), average waiting time in system or mean sojourn time (W) and exceedance probability of queue length ($\bar{F}_q$) or waiting time in queue ($\bar{F}_w$). Another important metric is traffic intensity (ρ) which is a measure of average occupancy of a server or in other words probability of the server being busy during a specified duration of time. It is expressed as the ratio of λ to μ.

Because of the stochastic nature of queueing systems, we generally assume completely specified distributions for the input variables in a queueing model namely interarrival time and service time. Consequently, distributions of output variables like number present in the system (system size), waiting time etc. are derived in terms of the given distributions of input variables. In real life, the assumption of a particular form for the distribution of an input variable may be justified from prior considerations, but numerical values of input parameters viz. arrival rate (λ) and service rate (μ) of these distributions are not given to us. So, estimation of these input parameters as well as corresponding QPM's which are non random functions of these input parameters is essential for a better decision making. Estimation in queueing theory is carried out by different researchers using both the maximum likelihood principle and the Bayesian method.

In the context of Bayesian estimation in Markovian or semi-Markovian queueing models, many researchers [1-2, 3, 5, 6, 9, 11, 13, 18, 19] estimate λ, μ and other measures of performance viz. traffic intnsity (ρ), no of customers in the system or avaerage system length ($L_s = \lambda/(\mu - \lambda)$) and mean sojourn time ($W = 1/(\mu - \lambda)$) observing arrival and service times of customers. There are other works [7, 14-16] where traffic intensity (ρ) is estimated recording the queue length of a system. While in the former situation, λ and μ are considered independently distributed and gamma distribution is taken as natural conjugate prior for the rate parameters, beta distribution is chosen as prior distribution for ρ in the later situation.

McGrath and Singpurwalla [11] suggest the use of Bivariate Normal Distribution as a joint prior in case of interdependence between λ and μ. Since both the parameters are positive quantites, for a suitable joint prior distribution for describing the stochastic relationship between the arrival and service rate parameters, they suggest the use of bivariate lognormal distribution.Their main emphasis is on establishing a qualitative relationship between the arrival and service processes, as an increase in the arrival pattern may tend to cause an increase in the service process. Therefore, they show the dependence using the correlation coefficient, without using the ergodicity condition $\lambda < \mu$.

We apply the ergodicity condition in the $M/M/1$ queueing model in this paper using beta-Stacy distribution (satisfying the natural restriction $\lambda < \mu$) as the joint prior for the Bayes estimation of queueing parameters. In section 2, It is shown that the maximum likelihood estimators (mle) of λ and μ do not exist, but that of ρ exists. Bayes estimators of QPM's along with corresponding risks [4] and 95% confidence intervals are obtained in section 3, using beta-Stacy distribution [10, 12] as the joint prior under both SELF and PLF [17]. A detailed simulation study is conducted in the last section. Sampling scheme of the present paper is to observe the states of a number of identical queues at any one selected point of time. Number of customers present in each of n iid $M/M/1/\infty/\infty$ queueing systems [8] at a given point in time under steady state is taken to constitute the sample.

2 Maximum Likelihood Estimation

Under the Markovian set up, inter-arrival and service time distributions are exponential and are given as $a(t) = \lambda exp(-\lambda t)$ and $b(t) = \mu exp(-\mu t); \lambda > 0, \mu > 0$. The steady state distribution of number of customers present in an $M/M/1$ queueing system [8] is given as

$$p_r = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^r; \quad \left(\frac{\lambda}{\mu} < 1\right) \tag{2.1}$$

Let $x_1, x_2, ..., x_n$ be a random sample of size n with x_i being number of customers present in the ith queue, $i = 1, 2, ..., n$.Thus the joint distribution of $x_1, x_2, ..., x_n$ can be written as

$$f(x_1, x_2, ..., x_n \mid \lambda, \mu) = \left(\frac{\lambda}{\mu}\right)^y \left(1 - \frac{\lambda}{\mu}\right)^n; \; y = \sum_{i=1}^{n} x_i \tag{2.2}$$

The likelihood function becomes

$$L(\lambda, \mu \mid x_1, x_2, ..., x_n) = \left(\frac{\lambda}{\mu}\right)^y \left(1 - \frac{\lambda}{\mu}\right)^n \tag{2.3}$$

with loglikelihood function given as

$$lnL(\lambda, \mu) = y \ln\left(\frac{\lambda}{\mu}\right) + n \ln\left(1 - \frac{\lambda}{\mu}\right). \tag{2.4}$$

Loglikelihood equations $\frac{\partial lnL}{\partial \lambda} = 0$ and $\frac{\partial lnL}{\partial \mu} = 0$ are not identifiable and hence do not yield mle's of λ and μ. Though, mle of ρ can be obtained solving both the equations and is computed as $\hat{\rho}_{mle} = \frac{y}{n+y}$. Now, $y = \sum_{i=1}^{n} x_i \sim NB(n, 1-\rho)$ with $E(y) = n\rho/(1-\rho)$ and $var(y) = n\rho/(1-\rho)^2$. $\hat{\rho}$ being a one-one function of y, takes values $\frac{y}{n+y}$, $y = 0(1)\infty$ with p.m.f

$$\begin{aligned} P\left(\frac{Y}{n+Y} = u\right) &= P\left(Y = \frac{nu}{1-u}\right) \\ &= \binom{\frac{nu}{1-u} + n - 1}{\frac{nu}{1-u}} (1-\rho)^n \rho^{\frac{nu}{1-u}}, \; u = \frac{y}{n+y} \end{aligned} \tag{2.5}$$

yielding,

$$E(\hat{\rho}) = \sum_{y=0}^{\infty} \left[\frac{y}{(n+y)} \binom{y+n-1}{y} (1-\rho)^n \rho^y\right] \tag{2.6}$$

Now, for large n,

$$\begin{aligned} E(\hat{\rho}) &\approx \frac{n\rho/(1-\rho)}{n + n\rho/(1-\rho)} \\ &= \rho \end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
Var(\hat{\rho}) &\simeq \left[(\frac{d\hat{\rho}_{mle}}{dy})^2 Var(Y) \right]_{E(Y)=\frac{n\rho}{1-\rho}} \\
&= \frac{n^2}{\{n + n\rho/(1-\rho)\}^4} \frac{n\ \rho}{(1-\rho)^2} \\
&= \frac{n^2(1-\rho)^4}{n} \ \frac{n\rho}{(1-\rho)^2} \\
&= \frac{\rho(1-\rho)^2}{n} \\
&\longrightarrow 0 \qquad as\ n \to \infty
\end{aligned} \tag{2.8}
$$

Therefore, $\hat{\rho}$ is a consistent estimator of ρ. Hence for large values of n,

$$
\frac{\sqrt{n}(\hat{\rho} - \rho)}{\sigma} \xrightarrow{D} N(0, 1) \tag{2.9}
$$

where

$$
\sigma^2 = \left[E(\frac{\delta}{\delta\rho} lnf(x_1, x_2, ..., x_n))^2 \right]^{-1} = \rho\,(1-\rho)^2 \tag{2.10}
$$

As mle of the input parameters and the other QPM's (except traffic intensity) do not exist in the current set up, we go for Bayesian estimation which is discussed in the next section.

3 Estimation Using Beta-Stacy Distribution as Prior

In this section, we derive Bayes estimator of λ, μ and their non random functions viz. traffic intnsity (ρ), average system length ($L_s = \lambda/(\mu - \lambda)$) and mean sojourn time ($W = 1/(\mu - \lambda)$) with one bivariate distribution viz. beta-Stacy taking observations on one random variable viz. number of customers present in the system (at some point of time) from n identical queueing systems. Thus our data correspond to system size and form an iid sample. Joint and Marginal posterior distributions of λ and μ are worked out in the next subsection and hence Bayes estimators of these queueing parameters along with their performance measures are obtained. The joint prior distribution of λ and μ, known as beta-Stacy is given as

$$
\tau(\lambda, \mu) = \frac{|\ c\ |}{\Gamma\alpha} \frac{1}{a^{\alpha c}} \frac{1}{B(\theta_1, \theta_2)} \mu^{\alpha c - \theta_1 - \theta_2} \lambda^{\theta_1 - 1} (\mu - \lambda)^{\theta_2 - 1} e^{-(\frac{\mu}{a})^c}\ ;\ 0 < \lambda < \mu\ ,\ \alpha, \theta_1, \theta_2 > 0, c \in \Re. \tag{3.1}
$$

3.1 Joint and marginal Posterior Distribution of (λ, μ)

Posterior distribution of (λ, μ) given the sample $(x_1, x_2, ..., x_n)$ works out as

$$
q(\lambda, \mu \mid data) = k_1 \mu^{\alpha c - \theta_1 - \theta_2 - y - n} \lambda^{\theta_1 + y - 1} (\mu - \lambda)^{n + \theta_2 - 1} e^{-(\mu/a)^c}, \tag{3.2}
$$

where the normalising constant k_1 is such that

$$k_1 \int_0^\infty \int_\mu^\infty q(\lambda, \mu \mid data) d\lambda d\mu = 1, \tag{3.3}$$

which gives

$$\begin{aligned} \frac{1}{k_1} &= \int_0^\infty e^{-\left(\frac{\mu}{a}\right)^c} \mu^{\alpha c - \theta_1 - \theta_2 - y - n} \left(\int_0^\mu \lambda^{\theta_1 + y - 1} (\mu - \lambda)^{n + \theta_2 - 1} d\lambda \right) d\mu \\ &= \frac{B(\theta_1 + y, n + \theta_2) \Gamma\alpha a^{\alpha c}}{\mid c \mid}. \end{aligned} \tag{3.4}$$

Marginal posterior density of λ and μ are given by

$$\begin{aligned} g_1(\lambda \mid data) &= \int_\lambda^\infty q(\lambda, \mu \mid data) d\mu \\ &= \frac{\mid c \mid \lambda^{\theta_1 + y - 1}}{B(\theta_1 + y, n + \theta_2)\Gamma\alpha a^{\alpha c}} \int_\lambda^\infty e^{-\left(\frac{\mu}{a}\right)^c} \mu^{\alpha c - \theta_1 - \theta_2 - y - n} (\mu - \lambda)^{n + \theta_2 - 1} d\mu \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} g_2(\mu \mid data) &= \int_0^\mu q(\lambda, \mu \mid data) d\lambda \\ &= \frac{\mid c^{-\left(\frac{\mu}{a}\right)^c} \mu^{\alpha c - \theta_1 - \theta_2 - y - n}}{B(\theta_1 + y, n + \theta_2)\Gamma\alpha a^{\alpha c}} \int_0^\mu \lambda^{\theta_1 + y - 1} (\mu - \lambda)^{n + \theta_2 - 1} d\lambda \\ &= \frac{\mid c \mid e^{-\left(\frac{\mu}{a}\right)^c} \mu^{\alpha c - 1}}{\Gamma\alpha}. \end{aligned} \tag{3.6}$$

3.2 Estimator under Squared Error Loss

Here we consider widely used loss function, the squared error loss function (SELF) which is symmetric and is given by

$$L_1(\widehat{\theta}_S^B) = (\widehat{\theta}_S^B - \theta)^2$$

where θ and $\widehat{\theta}_S^B$ are parameter or parametric function and estimator under SELF respectively. Minimizing $E_{\theta|data} L_1(\widehat{\theta}_S^B, \theta)$, i.e. solving $\frac{dE_{\theta|data} L_1(\widehat{\theta}_S^B, \theta)}{d\theta} = 0$, we get

$$\widehat{\theta}_S^B = E_{\theta|data}(\theta).$$

The Bayes estimator of λ, μ, ρ, L_s and W under SELF are derived as

$$\hat{\lambda}_S^B = k_1 \int_0^\infty \int_0^\infty e^{-\left(\frac{\lambda+\delta}{a}\right)^c} \lambda^{\theta_1 + y} (\lambda + \delta)^{\alpha c - \theta_1 - \theta_2 - y - n} \delta^{n + \theta_2 - 1} d\lambda d\delta; (\delta = \mu - \lambda) \tag{3.7}$$

$$\hat{\mu}_S^B = \frac{a\Gamma\left(\alpha + \frac{1}{c}\right)}{\Gamma\alpha} \tag{3.8}$$

$$\hat{\rho}_S^B = \frac{y + \theta_1}{y + \theta_1 + \theta_2 + n} \tag{3.9}$$

$$\hat{L_s}_S^B = \frac{y+\theta_1}{n+\theta_2-1}; n+\theta_2 > 1 \tag{3.10}$$

and

$$\hat{W}_S^B = \frac{\Gamma(\alpha-\frac{1}{c})(\theta_1+\theta_2+y+n-1)}{a\Gamma\alpha(n+\theta_2-1)}; \alpha > 1/c. \tag{3.11}$$

3.3 Estimator under Precautionary Loss Function

Most of the Bayes procedures are developed under the usual SELF which is symmetrical and give equal importance to the losses due to overestimation and underestimation of equal magnitude. There are situations where an underestimate is more serious than overestimate. In this case, use of symmetrical loss function might be inappropriate and a useful asymmetric loss function viz. precautionary loss function could be appropriate. This loss function is interesting in the sense that a slight modification of squared error loss introduces asymmetry. The Precautionary loss function (PLF) is given by

$$L_1(\widehat{\theta}_P^B) = \frac{(\widehat{\theta}_P^B - \theta)^2}{\widehat{\theta}_S^B}$$

where θ and $\widehat{\theta}_P^B$ are parameter or parametric function and estimator under PLF respectively. Minimizing $E_{\theta|data}L_1(\widehat{\theta}_P^B, \theta)$, i.e. solving $\frac{dE_{\theta|data}L_1(\widehat{\theta}_P^B,\theta)}{d\theta} = 0$, we get

$$\widehat{\theta}_P^B = \left(E_{\theta|data}(\theta^2)\right)^{1/2}.$$

The Bayes estimator of λ, μ, ρ, L_s and W under PLF are derived as

$$\hat{\lambda}_P^B = k_1 \int_0^\infty \int_0^\infty e^{-(\frac{\lambda+\delta}{a})^c} \lambda^{\theta_1+y+1} (\lambda+\delta)^{\alpha c-\theta_1-\theta_2-y-n} \delta^{n+\theta_2-1} d\lambda d\delta; (\delta = \mu - \lambda) \tag{3.12}$$

$$\hat{\mu}_P^B = \frac{a^{c+1}\Gamma\left(\alpha+\frac{1}{c}+1\right)}{\Gamma\alpha} \tag{3.13}$$

$$\hat{\rho}_P^B = \frac{(y+\theta_1+1)(y+\theta_1)}{(y+\theta_1+\theta_2+n+1)(y+\theta_1+\theta_2+n)} \tag{3.14}$$

$$\hat{L_s}_P^B = \frac{(y+\theta_1+1)(y+\theta_1)}{(n+\theta_2-1)(n+\theta_2-2)}; n+\theta_2 > 2. \tag{3.15}$$

The same Bayes estimator of ρ can be obtained using natural conjugate prior beta under both SELF and PLF. Therefore, the use of beta-Stacy distribution as prior in case of dependence is justified viz. one for its natural restriction and the other for the same Bayes estimator being obtained as in the case of beta prior.

In the same way, Bayes estimator of W is given by

$$\hat{W}_S^B = \frac{\Gamma(\alpha-\frac{2}{c})(\theta_1+\theta_2+y+n-1)(\theta_1+\theta_2+y+n-2)}{a^2\Gamma\alpha(n+\theta_2-1)(n+\theta_2-2)}; \alpha > 2/c,\ n+\theta_2 > 2. \tag{3.16}$$

4 Numerical Results

4.1 Simulation Procedure

In this section, Bayes estimates are obtained under SELF and PLF for two choices of λ and μ. For this purpose, a Monte-Carlo simulation has been carried out using R Software. The steps are given as follows:

- Given $(\lambda, \mu) = (1.2, 6)$ and $(4, 5)$, random samples of sizes $n = 5$, 20 and 50 are generated from (2.2) from where y is obtained.

- Scale parameters a and α are respectively chosen as 2 and $(2, 0.5)$ while shape parameter c as $(2, -2)$.

- The combinations of θ_1 and θ_2 are chosen as (4, 10, 15) and 3 respectively. As a special case of beta-Stacy distribution, McKays Bivarite Gamma [10] is also used as prior distribution choosing $\alpha = \theta_1 + \theta_2$ and $c = 1$. With these chosen combinations of hyper-parameters and sample size n, Bayes estimates (3.9-3.11) under SELF and (3.14-3.16) under PLF are obtained with generated sum y.

- Steps 1-3 are repeated N (=10,000) times and average of all these estimates are taken to yield the Bayes estimates as follows:

$$\hat{\theta^B} = \sum_{i=1}^{N} \hat{\theta_i^B}/N. \tag{4.1}$$

  The estimated risks corresponding to Bayes estimates under SELF and PLF are computed as

$$R(\theta_S^B, \theta) = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta_{S_i}^B} - \theta \right)^2 \tag{4.2}$$

  and

$$R(\theta_P^B, \theta) = \frac{1}{N} \sum_{i=1}^{N} \frac{\left(\hat{\theta_{P_i}^B} - \theta \right)^2}{\hat{\theta_{P_i}^B}} \tag{4.3}$$

  respectively.

- 95% confidence interval of each of the parameters for the stated choices of hyper-parameters and n is also obtained along with the estimates and the risks. For calculation of equal-tailed confidence limits, 10,000 estimates of any parameter are arranged in increasing order. Then from the ordered estimates, 10,000*0.025 = 250th and 10,000*0.975 = 9750th estimates are taken as lower confidence limit and upper confidence limit respectively.

Maximum likelihood estimates of ρ are shown in Table 1. Bayes estimates of the QPM's are shown in Table 2 and Table 3 under SELF and PLF along with the estimated risks (in ( )) and 95% confidence interval (in [,]).

Table 1. Maximum likelihood estimates of ρ

(λ, μ)	n	$\widehat{\rho}_{mle}$	MSE
1.2, 6	5	0.1737	0.0219
	20	0.1912	0.0062
	50	0.1963	0.0026
4, 5	5	0.7666	0.0118
	20	0.7918	0.0019
	50	0.7970	0.0007

Insert Table 2 and Table 3 here

4.2 Simulation Results

While Bayes estimates of average waiting time in the system depends on all the stated hyper-parameters, those of traffic intensity and average system length depend on θ_1 and θ_2 only. Observations from Tables 1, 2 and 3 are stated as follows:

- The ml estimates of ρ are very stable around the true value and corresponding errors also diminish with large sample size.

- For given θ_1, θ_2, a, c, α and n, Bayes estimates of all queueing parameters under SELF perform better than those under PLF in terms of estimated risks. There is only one exception to it. Average waiting time under PLF performs better than that under SELF for $\theta_1 < \theta_2$ and for some choices of λ and μ such that $\frac{\lambda}{\mu} < k$ where $0 < k < 0.4$.

- Bayes estimates of ρ under PLF are pretty unstable even for large sample size.

- For fixed values of other hyper-parameters, as $(\theta_1 - \theta_2)$ increases, Bayes estimates under both SELF and PLF also increase. On the other hand, under the same set-up, as $(\theta_2 - \theta_1)$ increases, Bayes estimates decrease.

- As expected, the risks associated with all the estimates diminish with increase in the sample size (number of independent queues observed) for both SELF and LLF except for a few sampling fluctuations..

5 Conclusion

Although Bayes estimation of QPM's are carried out on different occasions using independent gamma priors or bivariate normal distribution as prior, the condition of ergodicity ($\lambda < \mu$) is never used in any of the work. We have used this ergodicity condition in the form of using beta-Stacy distribution as prior which has a natural restricted range ($0 < \lambda < \mu < \infty$). Bayes estimates of traffic intensity, average system length and average waiting time in the system are obtained using the stated prior under both SELF and PLF. It is observed from the simulation study that the use of beta-Stacy as prior is justified as it yields stable estimates of the QPM's, specially under SELF.

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Table 2: Different Estimates and their estimated risks for $\lambda = 1.2, \mu = 6, \rho = 0.2$

n=5						
Hyper-parameters	ρ_S^B	L_S^B	W_S^B	ρ_P^B	L_P^B	W_P^B
θ_1=4, θ_2=3, α=2, c=2	0.3920 (0.0397) [0.3333, 0.5000]	0.7523 (0.2845) [0.5714, 1.1249]	0.7765 (0.3291) [0.6963, 0.9495]	0.1730 (0.0172) [0.1282, 0.2647]	0.8232 (0.4129) [0.4762, 1.7143]	0.8320 (0.4240) [0.6548, 1.2500]
θ_1=3, θ_2=4, α=2, c=2	0.3156 (0.0169) [0.2500, 0.4375]	0.5325 (0.1045) [0.3750, 0.8750]	0.6791 (0.2264) [0.6093, 0.8308]	0.1181 (0.0934) [0.0769, 0.2059]	0.4284 (0.1146) [0.2143, 1.000]	0.6233 (0.1921) [0.4911, 0.9375]
θ_1=10, θ_2=3, α=2, c=2	0.5830 (0.1474) [0.5556, 0.6363]	1.6092 (1.8806) [1.4286, 2.000]	1.1562 (0.9049) [1.0761, 1.3293]	0.3526 (0.0668) [0.3216, 0.4150]	3.3280 (2.8476) [2.6190, 5.000]	1.8866 (2.8961) [1.6190, 2.5000]
θ_1=3, θ_2=10, α=2, c=2	0.2186 (0.0027) [0.1667, 0.3182]	0.3053 (0.0113) [0.2143, 0.5000]	0.5784 (0.1386) [0.5381, 0.6647]	0.0584 (0.4456) [0.0351, 0.1107]	0.1327 (0.2348) [0.0659, 0.3077]	0.4359 (0.0559) [0.3736, 0.5769]
θ_1=15, θ_2=3, α=2, c=2	0.6690 (0.2202) [0.6522, 0.7037]	2.3183 (4.3083) [2.1429, 2.7143]	1.4704 (1.5987) [1.3926, 1.6459]	0.4566 (0.1444) [0.4348, 0.5026]	6.6921 (6.2017) [5.7143, 9.0476]	3.0822 (8.3751) [2.7500, 3.8690]
θ_1=3, θ_2=15, α=2, c=2	0.1730 (0.0023) [0.1304, 0.2593]	0.2233 (0.0050) [0.1579, 0.3684]	0.5421 (0.1122) [0.5131, 0.6064]	0.0371 (0.8939) [0.0217, 0.0741]	0.0696 (0.7103) [0.0351, 0.1637]	0.3791 (0.0310) [0.3377, 0.4752]
θ_1=4, θ_2=3, α=2, c=-2	0.3920 (0.0397) [0.3333, 0.5000]	0.7523 (0.2845) [0.5714, 1.1249]	1.1623 (0.9241) [1.0445, 1.4243]	0.1730 (0.0172) [0.1282, 0.2647]	0.8232 (0.4129) [0.4762, 1.7143]	1.6565 (2.2341) [1.3095, 2.5000]
θ_1=4, θ_2=3, α=7, c=1	0.3920 (0.0397) [0.3333, 0.5000]	0.7523 (0.2845) [0.5714, 1.1249]	0.1460 (0.0041) [0.1310, 0.1786]	0.1730 (0.0172) [0.1282, 0.2647]	0.8232 (0.4129) [0.4762, 1.7143]	0.0278 (0.0327) [0.0218, 0.0417]
θ_1=4, θ_2=3, α=0.5, c=5	0.3920 (0.0397) [0.3333, 0.5000]	0.7523 (0.2845) [0.5714, 1.1249]	1.4760 (1.6290) [1.3261, 1.8084]	0.1730 (0.0172) [0.1282, 0.2647]	0.8232 (0.4129) [0.4762, 1.7143]	4.4462 (18.9269) [3.5144, 6.7093]
n=20						
θ_1=4, θ_2=3, α=2, c=2	0.2762 (0.0088) [0.1786, 0.3947]	0.4075 (0.0376) [0.2272, 0.6818]	0.6237 (0.1750) [0.5438, 0.7452]	0.0852 (0.2269) [0.0369, 0.1619]	0.2068 (0.1059) [0.0649, 0.5195]	0.5054 (0.0954) [0.3799, 0.7208]
θ_1=3, θ_2=4, α=2, c=2	0.2453 (0.0053) [0.1429, 0.3684]	0.3473 (0.0212) [0.1739, 0.6087]	0.5970 (0.1534) [0.5202, 0.7128]	0.0689 (0.3804) [0.0246, 0.1417]	0.1541 (0.2345) [0.0395, 0.4150]	0.4622 (0.0704) [0.3468, 0.6581]
θ_1=10, θ_2=3, α=2, c=2	0.3929 (0.0387) [0.3235, 0.4773]	0.6838 (0.2012) [0.5000, 0.9545]	0.7461 (0.2918) [0.6647, 0.8661]	0.1620 (0.0178) [0.1109, 0.2333]	0.5360 (0.1649) [0.2857, 1.000]	0.7259 (0.2782) [0.5714, 0.9773]
θ_1=3, θ_2=10, α=2, c=2	0.2079 (0.0026) [0.1176, 0.3023]	0.2769 (0.0081) [0.1379, 0.4483]	0.5658 (0.1293) [0.5042, 0.6417]	0.0499 (0.6514) [0.0168, 0.0962]	0.0970 (0.5169) [0.0246, 0.2241]	0.4127 (0.0450) [0.3251, 0.5302]
θ_1=15, θ_2=3, α=2, c=2	0.4634 (0.0703)	0.9093 (0.4479)	0.8460 (0.4093)	0.2213 (0.0050)	0.9234 (0.4951)	0.9355 (0.5422)

	[0.4103, 0.5306]	[0.7273, 0.1818]	[0.7654, 0.9668]	[0.1744, 0.2865]	[0.5887, 1.5195]	[0.7608, 1.2208]
$\theta_1=3, \theta_2=15, \alpha=2, c=2$	0.1838 (0.0024) [0.1026, 0.2857]	0.2361 (0.0056) [0.1176, 0.4118]	0.5477 (0.1162) [0.4952, 0.6256]	0.0393 (0.9336) [0.0128, 0.0857]	0.0702 (0.8484) [0.0178, 0.1872]	0.3856 (0.0336) [0.3133, 0.5027]
$\theta_1=4, \theta_2=3, \alpha=2, c=-2$	0.2762 (0.0088) [0.1786, 0.3947]	0.4075 (0.0376) [0.2272, 0.6818]	0.9356 (0.5347) [0.8157, 1.1179]	0.0852 (0.2269) [0.0369, 0.1619]	0.2068 (0.1059) [0.0649, 0.5195]	1.011 (0.6739) [0.7597, 1.4416]
$\theta_1=4, \theta_2=3, \alpha=7, c=1$	0.2762 (0.0088) [0.1786, 0.3947]	0.4075 (0.0376) [0.2272, 0.6818]	0.4101 (0.0385) [0.2273, 0.6818]	0.0852 (0.2269) [0.0369, 0.1619]	0.2068 (0.1059) [0.0649, 0.5195]	0.0169 (0.0367) [0.0127, 0.0240]
$\theta_1=4, \theta_2=3, \alpha=0.5, c=5$	0.2762 (0.0088) [0.1786, 0.3947]	0.4075 (0.0376) [0.2272, 0.6818]	1.1881 (0.9691) [1.0357, 1.4193]	0.0852 (0.2269) [0.0369, 0.1619]	0.2068 (0.1059) [0.0649, 0.5195]	2.7145 (6.4877) [2.0389, 3.8687]
n=50						
$\theta_1=4, \theta_2=3, \alpha=2, c=2$	0.2345 (0.0031) [0.1452, 0.3205]	0.3165 (0.0103) [0.1731, 0.4808]	0.5834 (0.1418) [0.5198, 0.6561]	0.0594 (0.4324) [0.0230, 0.1055]	0.1143 (0.3044) [0.0339, 0.2451]	0.4368 (0.0549) [0.3450, 0.5517]
$\theta_1=3, \theta_2=4, \alpha=2, c=2$	0.2206 (0.0023) [0.1429, 0.3077]	0.2925 (0.0074) [0.1698, 0.4528]	0.5727 (0.1339) [0.5184, 0.6438]	0.0530 (0.5385) [0.0223, 0.0974]	0.0985 (0.4214) [0.0327, 0.2177]	0.4209 (0.0476) [0.3431, 0.5308]
$\theta_1=10, \theta_2=3, \alpha=2, c=2$	0.2960 (0.0105) [0.2319, 0.3690]	0.4324 (0.0389) [0.3077, 0.5962]	0.6347 (0.1829) [0.5795, 0.7073]	0.0916 (0.1541) [0.0563, 0.1389]	0.2049 (0.0478) [0.1026, 0.3741]	0.5174 (0.0986) [0.4295, 0.6416]
$\theta_1=3, \theta_2=10, \alpha=2, c=2$	0.2029 (0.0017) [0.1304, 0.2857]	0.2622 (0.0045) [0.1525, 0.4068]	0.5593 (0.1240) [0.5107, 0.6234]	0.0449 (0.6948) [0.0186, 0.0840]	0.0789 (0.6024) [0.0263, 0.1753]	0.4008 (0.0389) [0.3328, 0.4972]
$\theta_1=15, \theta_2=3, \alpha=2, c=2$	0.3404 (0.0207) [0.2838, 0.4045]	0.5297 (0.0841) [0.4038, 0.6923]	0.6779 (0.2216) [0.6221, 0.7499]	0.1197 (0.0656) [0.0832, 0.1663]	0.3024 (0.0262) [0.1742, 0.5023]	0.5905 (0.1496) [0.4955, 0.7217]
$\theta_1=3, \theta_2=15, \alpha=2, c=2$	0.1911 (0.0017) [0.1096, 0.2697]	0.2430 (0.0040) [0.1250, 0.3750]	0.5508 (0.1180) [0.4985, 0.6093]	0.0400 (0.8350) [0.0133, 0.0749]	0.0679 (0.7731) [0.0179, 0.1488]	0.3885 (0.0340) [0.3170, 0.4747]
$\theta_1=4, \theta_2=3, \alpha=2, c=-2$	0.2345 (0.0031) [0.1452, 0.3205]	0.3165 (0.0103) [0.1731, 0.4808]	0.8752 (0.4472) [0.7925, 0.9842]	0.0594 (0.4324) [0.0230, 0.1055]	0.1143 (0.3044) [0.0339, 0.2451]	0.8739 (0.4534) [0.7130, 1.1033]
$\theta_1=4, \theta_2=3, \alpha=7, c=1$	0.2345 (0.0031) [0.1452, 0.3205]	0.3165 (0.0103) [0.1731, 0.4808]	0.1097 (0.0098) [0.7925, 0.9842]	0.0594 (0.4324) [0.0994, 0.1234]	0.1143 (0.3044) [0.0339, 0.2451]	0.0145 (0.0376) [0.0119, 0.0184]
$\theta_1=4, \theta_2=3, \alpha=0.5, c=5$	0.2345 (0.0031) [0.1452, 0.3205]	0.3165 (0.0103) [0.1731, 0.4808]	1.1117 (0.8202) [1.0062, 1.2496]	0.0594 (0.4324) [0.0994, 0.1234]	0.1143 (0.3044) [0.0339, 0.2451]	2.3475 (4.6522) [1.9136, 2.9610]

Table 3: Different Estimates and their estimated risks for $\lambda = 4, \mu = 5, \rho = 0.8$

n=5						
Hyper-parameters	ρ_S^B	L_S^B	W_S^B	ρ_P^B	L_P^B	W_P^B
θ_1=4, θ_2=3, α=2, c=2	0.7276 (0.0124) [0.5294, 0.8571]	3.4676 (2.4104) [1.2857, 6.8571]	1.9796 (1.3773) [1.0128, 3.4816]	0.5431 (0.1902) [0.2941, 0.7368]	17.0875 (10.97) [2.1428, 56.000]	6.2557 (47.5842) [1.4286, 17.6786]
θ_1=3, θ_2=4, α=2, c=2	0.6920 (0.0208) [0.4706, 0.8393]	2.8898 (2.8374) [1.000, 5.8750]	1.7236 (0.8387) [0.8862, 3.0464]	0.4951 (0.3011) [0.2353, 0.7068]	11.7908 (6.769) [1.2857, 40.280]	4.6426 (23.9552) 1.0714, 13.2589]
θ_1=10, θ_2=3, α=2, c=2	0.7744 (0.0039) [0.6522, 0.8689]	4.2604 (2.0841) [2.1429, 7.5714]	2.3310 (2.1674) [1.3926, 3.7981]	0.6078 (0.0833) [0.4348, 0.7567]	24.2390 (17.23) [5.7143, 68.142]	8.4400 (79.2515) [2.7500, 21.0714]
θ_1=3, θ_2=10, α=2, c=2	0.5794 (0.0601) [0.3478, 0.7581]	1.6491 (6.0491) [0.5714, 3.3571]	1.1738 (0.1328) [0.6963, 1.9307]	0.3535 (0.8684) [0.1304, 0.5776]	3.6179 (5.3843) [0.3956, 12.395]	1.9790 (2.3161) [0.6346, 5.0275]
θ_1=15, θ_2=3, α=2, c=2	0.8047 (0.0019) [0.7143, 0.8806]	5.0218 (3.1718) [2.8571, 8.4286]	2.6683 (3.2011) [1.7092, 4.1780]	0.6533 (0.0443) [0.5172, 0.7770]	32.7407 (25.41) [10.000, 84.286]	10.9461 (131.255) [4.1786, 25.5357]
θ_1=3, θ_2=15, α=2, c=2	0.5136 (0.0936) [0.2857, 0.7015]	1.2238 (7.9890) [0.4211, 2.4737]	0.9854 (0.0556) [0.6297, 1.5392]	0.2810 (1.4589) [0.0887, 0.4952]	1.9466 (11.696) [0.2105, 6.5965]	1.3486 (0.5981) [0.5132, 3.1360]
θ_1=4, θ_2=3, α=2, c=-2	0.7276 (0.0124) [0.5294, 0.8571]	3.4676 (2.4104) [1.2857, 6.8571]	2.9472 (4.6893) [1.5193, 5.2224]	0.5431 (0.1902) [0.2941, 0.7368]	17.0875 (10.97) [2.1428, 56.000]	12.2848 (198.67) [2.8571, 35.3571]
θ_1=4, θ_2=3, α=7, c=1	0.7276 (0.0124) [0.5294, 0.8571]	3.4676 (2.4104) [1.2857, 6.8571]	0.3677 (0.4138) [0.1905, 0.6429]	0.5431 (0.1902) [0.2941, 0.7368]	17.0875 (10.97) [2.1428, 56.000]	0.2028 (0.6553) [0.0476, 0.5679]
θ_1=4, θ_2=3, α=0.5, c=5	0.7276 (0.0124) [0.5294, 0.8571]	3.4676 (2.4104) [1.2857, 6.8571]	3.7494 (9.0355) [1.9289, 6.6307]	0.5431 (0.1902) [0.2941, 0.7368]	17.0875 (10.97) [2.1428, 56.000]	33.1537 (1569.76) [7.6677, 94.8884]
n=20						
θ_1=4, θ_2=3, α=2, c=2	0.7774 (0.0023) [0.6849, 0.8477]	3.8191 (0.8608) 0.2727, 5.8182]	2.1354 (1.4518) [1.4502, 3.0212]	0.6079 (0.0736) [0.4720, 0.7194]	16.3296 (9.566) [5.5195, 35.740]	6.2419 (33.2507) [2.7662, 12.0942]
θ_1=3, θ_2=4, α=2, c=2	0.7678 (0.0030) [0.6712, 0.8389]	3.6069 (0.8896) [2.1304, 5.4348]	2.0414 (1.2288) [1.3871, 2.8513]	0.5932 (0.0868) [0.4535, 0.7047]	14.5338 (7.925) [4.8419, 31.126]	5.6869 (26.6020) [2.5257, 10.7490]
θ_1=10, θ_2=3, α=2, c=2	0.7901 (0.0015) [0.7051, 0.8526]	4.0942 (0.8399) [2.5000, 6.0455]	2.2573 (1.7440) [1.5509, 3.1220]	0.6272 (0.0572) [0.4998, 0.7277]	18.6260 (11.68) [6.6667, 38.576]	6.9536 (41.8902) [3.1667, 12.9167]
θ_1=3, θ_2=10, α=2, c=2	0.7264 (0.0078) [0.6154, 0.8077]	2.8671 (1.7603) [1.6552, 4.3448]	1.7135 (0.6028) [1.1765, 2.3684]	0.5319 (0.1582) [0.3817, 0.6534]	9.1098 (3.3421) [2.8966, 19.707]	3.9610 (10.8453) [1.8017, 7.3491]

$\theta_1=15, \theta_2=3, \alpha=2, c=2$	0.7995 (0.0012) [0.7229, 0.8571]	4.3199 (0.9278) [2.7273, 6.2727]	2.3573 (2.0044) [1.6516, 3.2226]	0.6418 (0.0464) [0.5250, 0.7354]	20.6206 (13.55) [7.9221, 41.519]	7.5651 (50.0331) [3.5942, 13.7662]
$\theta_1=3, \theta_2=15, \alpha=2, c=2$	0.6946 (0.0138) [0.5783, 0.7826]	2.4403 (2.7833) [1.4118, 3.7059]	1.5244 (0.3439) [1.0689, 2.0852]	0.4871 (0.2323) [0.3373, 0.6135]	6.5708 (1.6673) [2.0963, 14.262]	3.1129 (5.6622) [1.4799, 5.6684]
$\theta_1=4, \theta_2=3, \alpha=2, c=-2$	0.7774 (0.0023) [0.6849, 0.8477]	3.8191 (0.8608) 0.2727, 5.8182]	3.2114 (5.2611) [2.1451, 4.5318]	0.6079 (0.0736) [0.4720, 0.7194]	16.3296 (9.566) [5.5195, 35.740]	12.5524 (157.187) [5.3788, 24.1883]
$\theta_1=4, \theta_2=3, \alpha=7, c=1$	0.7774 (0.0023) [0.6849, 0.8477]	3.8191 (0.8608) 0.2727, 5.8182]	0.4028 (0.3623) [0.2689, 0.5644]	0.6079 (0.0736) [0.4720, 0.7194]	16.3296 (9.566) [5.5195, 35.740]	0.2092 (0.6317) [0.0896, 0.3978]
$\theta_1=4, \theta_2=3, \alpha=0.5, c=5$	0.7774 (0.0023) [0.6849, 0.8477]	3.8191 (0.8608) 0.2727, 5.8182]	4.0665 (9.9976) [2.7619, 5.7539]	0.6079 (0.0736) [0.4720, 0.7194]	16.3296 (9.566) [5.5195, 35.740]	33.5056 (1225.36) [14.8475, 64.914]
n=50						
$\theta_1=4, \theta_2=3, \alpha=2, c=2$	0.7908 (0.0008) [0.7337, 0.8364]	3.9288 (0.3782) [2.8077, 5.2115]	2.1840 (1.4751) [1.6872, 2.7524]	0.6268 (0.0524) [0.5392, 0.7000]	16.1953 (9.285) [8.0828, 27.795]	6.2632 (30.1505) [3.6770, 9.8045]
$\theta_1=3, \theta_2=4, \alpha=2, c=2$	0.7865 (0.0009) 0.73, 0.8334)	3.8257 (0.3889) [2.7547, 5.0943]	2.1384 (1.3662) [1.6638, 2.7005]	0.6199 (0.0570) [0.5339, 0.6949]	15.3568 (8.506) 8.0928, 27.795]	6.0021 (27.2967) [3.5742, 9.4345]
$\theta_1=10, \theta_2=3, \alpha=2, c=2$	0.7953 (0.0006) [0.7427, 0.8384]	4.0307 (0.3643) [2.9423, 5.2885]	2.2292 (1.5822) [1.7469, 2.7865]	0.6337 (0.0476) [0.5526, 0.7034]	17.0147 (10.04) [8.8846, 28.621]	6.5190 (32.9407) [3.9423, 10.0492]
$\theta_1=3, \theta_2=10, \alpha=2, c=2$	0.7687 (0.0018) [0.7073, 0.8182]	3.4446 (0.5999) [2.4576, 4.5763]	1.9694 (0.9970) [1.5321, 2.4709]	0.5924 (0.0787) [0.5013, 0.6699]	12.4254 (5.848) [6.1864, 21.382]	5.0786 (18.1792) [3.0254, 7.8837]
$\theta_1=15, \theta_2=3, \alpha=2, c=2$	0.7991 (0.0006) [0.7476, 0.8408]	4.1264 (0.3898) [3.0192, 5.3846]	2.2716 (1.6903) [1.7810, 2.8291]	0.6398 (0.0438) [0.5598, 0.7074]	17.8231 (10.80) [9.3537, 39.668]	6.7690 (35.9262) [4.0980, 10.3594]
$\theta_1=3, \theta_2=15, \alpha=2, c=2$	0.7542 (0.0029) [0.6905, 0.8054]	3.1742 (0.9266) [2.2656, 4.2031]	1.8496 (0.7699) [1.4470, 2.3056]	0.5704 (0.0993) [0.4778, 0.6491]	10.5342 (4.210) [5.2505, 18.013]	4.4706 (13.1849) [2.6954, 6.8549]
$\theta_1=4, \theta_2=3, \alpha=2, c=-2$	0.7908 (0.0008) [0.7337, 0.8364]	3.9288 (0.3782) [2.8077, 5.2115]	3.2790 (5.3602) [2.5309, 4.1286]	0.6268 (0.0524) [0.5392, 0.7000]	16.1953 (9.285) [8.0828, 27.795]	12.5508 (143.369) [7.3541, 19.6090]
$\theta_1=4, \theta_2=3, \alpha=7, c=1$	0.7908 (0.0008) [0.7337, 0.8364]	3.9288 (0.3782) [2.8077, 5.2115]	0.4111 (0.3494) [0.3173, 0.5176]	0.6268 (0.0524) [0.5392, 0.7000]	16.1953 (9.285) [8.0828, 27.795]	0.2091 (0.6282) [0.1226, 0.3268]
$\theta_1=4, \theta_2=3, \alpha=0.5, c=5$	0.7908 (0.0008) [0.7337, 0.8364]	3.9288 (0.3782) [2.8077, 5.2115]	4.1581 (10.229) [3.2296, 5.2257]	0.6268 (0.0524) [0.5392, 0.7000]	16.1953 (9.285) [8.0828, 27.795]	33.5762 (1128.91) [19.9366, 52.299]

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<tr><td colspan="2">Abstract:

Bayes estimators of different queueing performance measures are derived in steady state by recording system size from each of n iid M/M/1 queues. The Bayes estimators are obtained under both squared error loss function and precautionary loss function with a bivariate distribution, Beta-Stacy as prior with natural restriction $0 < \lambda < \mu$ where λ and μ are arrival rate and service rate respectively. A comprehensive simulation results are also shown at the last section.</td></tr>
<tr><td colspan="2">Key Words/Phrases: Bayes estimator; M/M/1 queue; Queue length; Beta-Stacy distribution; Monte Carlo simulation.</td></tr>
<tr><td colspan="2">Referencing Style Followed: APA</td></tr>
</table>

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