Dynamic Feedback Effect And Skewness In Non-Stationary Stochastic Volatility Model With Leverage

Sujay K Mukhoti*

Indian Institute of Management, Kozhikode

IIMK PO, Kozhikode; Kerala- 673570, India

Abstract

In this paper I present a new single factor model for assets return observed in discrete time and its latent volatility with a common "market factor". This model attempts to unify the concept of feedback effect and skewness in return distribution. Further, it generalizes existing stochastic volatility model with constant feedback to a framework with time varying feedback. As an immediate consequence dynamic skewness and leverage effect follows. However, the dynamic structure violates weak-stationarity assumption usually considered for the heteroskedastic models for returns and hence the concept of bounded stationarity is introduced to address the issue of non-stationarity. The single factor model also helps to reduce the number of parameters to be estimated compared to existing SV models with separate feedback and skewness parameters. A characterization of the error distributions for returns and volatility is provided on the basis of existence of conditional moments. Finally, an application of the model has been explained with Normal error and half Normal market factor distribution.

*e-mail:sujay.mukhoti@gmail.com

1 Introduction

Research in financial econometrics has seen a surge in the area of time-varying volatility models for asset returns over last three decades. Stochastic volatility (SV) model (Taylor 1982) has been one of the key instruments to address such an issue. In addition SV model explains some interesting aspects of asset returns observed empirically and known as "stylized facts". Some of the important stylized facts are volatility clustering indicating similar volatility periods clustering together and the negative relation between the return and its volatility divulging their movement in opposite direction. The work of Taylor (1982) models time varying volatility of financial returns as a latent auto-regressive process to account for the volatility clustering. Since then multitude of SV models have been developed to explain different stylized facts about asset returns. A comprehensive review of the SV models can be obtained from the works of Shephard & Andersen (2009) and Chib, Omori & Asai (2009).

Recent works in the SV literature emphasizes on three important aspects of such models for asset returns observed in discrete time *viz.* the negative correlation between current asset return and its future volatility (or leverage effect), return skewness and the correlation between current volatility and future returns (or the feedback effect). SV model and variants of the same has been developed to explain the time varying volatility of asset return and some of the three aspects mentioned above. Renault (2009) provides a comprehensive account of leverage, feedback and points out towards the possibility of a connection between the feedback effect and return skewness. However, no work has been done so far to explain the connection between feedback effect and return skewness along with volatility clustering and leverage for discrete time asset returns to the best of the knowledge of the author. In this paper, I develop a parsimonious generalized SV model to explain the relation between conditional feedback and return skewness in presence of leverage and extend it to an SV model with time dependent feedback, skewness and leverage parameters.

The time series data on asset returns provides evidence of correlation between asset return and its volatility (Nelson 1991) and skewness in asset returns (Harvey & Siddique 1999). The fact that a decline in current price would lead to increase in future volatility, could be attributed to changes in financial leverage (Nelson 1991) and such a correlation is called the leverage effect. On the other hand, the negative correlation between current volatility and future return is known as the feedback effect. Feedback effect may be attributed to the fact that an anticipated increase in volatility results in immediate price fall (French, Schwert & Stambaugh 1987). Bollerslev, Litvinova & Tauchen (2006) shows that a stronger signal is reflected through the contemporaneous correlation between asset return and its volatility and concludes in favor of the contemporaneous correlation as a measure of volatility feedback effect. Jacquier, Polson & Rossi (2004) used such an SV model to develop a Monte Carlo Markov Chain method for estimation of model parameters. However, Renault (2009) points out an alternative explanation to the negative contemporaneous correlation using the return skewness. The intuition behind such possibility could be justified by the following argument. The magnitude of volatility increase due to price fall is much higher than the magnitude by which volatility decreases in case of price increase, which is known as volatility asymmetry. Thus the conditional volatility for negative returns is more compared to the same for positive returns. This fact leads to the skewness in the innovation distribution. Tsiotas (2011) and references therein provide an account of SV models developed so far with leverage and skewed shocks. Recently, Feunou & Tédongap (2012) developed an affine SV model (denoted as SVS model) with standardized inverse Gaussian return shocks and autoregressive Gamma latent factors which accommodates both the feedback effect (mentioned as leverage by the authors) and the conditional skewness in asset returns. However, the SVS model does not provide explicit form of the volatility process and hence the intensity of volatility clustering is difficult to observe directly.

In this paper I propose a parsimonious representation of such an interlocked explication of feedback effect and skewness in presence of leverage effect. Both of them could be looked upon as a resultant of the influence of a common positive stochastic factor to the symmetric return and volatility shocks. A financial justification of the presence of such a stochastic factor could be in assuming the presence of a positive "market factor" which impacts both return and volatility with different magnitudes. Direction of market factor impacts could be similar or opposite. This mechanism generates perturbation to symmetric random shocks by a positive random variable and generates asymmetry in returns whereas the common factor generates the feedback effect. Further, Bollerslev, Sizova & Tauchen (2012) provides empirical evidence of the dynamic nature of the correlation between return and volatility. On the other hand Harvey & Siddique (1999) and recently Boyer, Mitton & Vorkink (2010) provide evidences of time dependent conditional return skewness in asset returns.

Based on the above findings I assume the weights of the market factor on return and volatility to be time varying so that the feedback effect and conditional skewness are dynamic. Individual impact of the stochastic market factor on return and volatility are measured by the corresponding time dependent coefficients which will be referred to as impact parameters here onwards. The underlying reason of different directions and magnitudes of the time varying conditional skewness and the feedback effect could be then comprehended in terms of the impact parameters.

The main complexity of this proposed model is that it violates weak-stationarity condition of the volatility process. Weak stationarity is crucial to a stochastic process as it restricts the process to increase indefinitely in expectation with time. In this paper I introduce the concept of bounded stationarity in terms of 1^{st} and 2^{nd} order moments of a stochastic process to relax the existing weak-stationarity condition yet ensuring that the process does not explode. I also provide here a characterization of the auto-regressive process of order one, which is most commonly used to describe volatility process in SV models, in the light of bounded stationarity in this paper.

The proposed model is developed under general distributions for return, volatility and the market factor. Many of the existing SV models has been shown to be particular cases of this generalized SV model. An immediate characterization of the plausible distributions for return, volatility and market factor has been given based on the existence of return moments and the feedback effect. Further, I provide explanation of the volatility asymmetry in terms of the market factor influence assuming the usual Gaussian framework for both return and volatility along with a standard half normal market factor. Such affine combination of Normal and Half-Normal distribution results in a variant of a general class of distributions containing standard Normal known as skew-normal distribution (Azzalini & Dalla Valle 1996). Since the proposed model uses only the conditional skewness parameters and feedback results in from the single factor model structure, the number of parameters to be estimated is reduced compared to the existing SV models (dynamic as well as static) which use separate parameters for feedback and skewness. Moreover, in order to extend the existing SV models with separate parameters for conditional skewness and feedback to a dynamic framework,large number of parameters will be required compared to number of available observations resulting into a highly saturated SV model. The model developed in this paper becomes advantageous in such a case as the number of parameters to be estimated is significantly less compared to the above mentioned case.

The paper is organized as follows. In section (2). we introduce the general framework for joint model with time dependent impact parameters and show some of the existing and well known SV models as special cases of the proposed joint model. The concept of bounded stationarity is introduced in this section to tackle non-stationarity in dynamic feedback SV model. Section (3) presents an example of generalized SVDF model with half-normal and Gaussian distributions. The expression for the dynamic feedback and leverage are presented here and necessary and sufficient conditions for negative feedback has been discussed. Section (4) contains discussion on the model and its application and some direction for the future work.

2 Dynamic Feedback SV Model with Common Market Factor

Let P_t be the daily price of an asset and $\log \frac{P_t}{P_{t-1}}$ be the log return. The time series of mean-corrected daily log returns is denoted by y_t and the underlying latent volatilities by θ_t. Let us start with the SV model proposed by Jacquier, Polson & Rossi (1994) which is given as follows:

$$y_t = e^{\frac{\theta_t}{2}}\epsilon_t, \quad (2.1)$$

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + \eta_t, t = 1, ..., T \quad (2.2)$$

ϵ_t and η_t being independent sequences of independently and identically distributed (iid) random shocks (or innovations) with both the means 0 and variances 1 and σ^2 respectively. ϕ is the volatility clustering parameter which reflects the stylized fact that volatility pattern (high or low) cluster together. Subsequently SV models with contemporaneous correlation (ρ) between ϵ_t and η_t has been developed by Jacquier et al. (2004). Such SV model with the feedback effect ρ relates the changes in volatility to the sign and magnitude of price changes which helps in pricing the options more accurately.

In this paper, I consider a new SV model with independent symmetric random shocks ϵ_t and η_t and a general positive common factor or "market factor", say γ_t, which impacts the return and its latent volatility at each time point. However, such impacts on return and its volatility may be different in magnitude and direction and may vary over time (Boyer et al. 2010). Let $\lambda_{y,t} \in \mathcal{R}$ and $\lambda_{\theta,t} \in \mathcal{R}$ denote the dynamic impacts of the market factor on the return and its latent volatility respectively. Thus the new SV model with a common dynamic market factor is given as

$$y_t = \mu_{y,t} + e^{\frac{\theta_t}{2}}(\lambda_{y,t}\gamma_t + \epsilon_t) \quad (2.3)$$

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + \mu_{\theta,t} + (\lambda_{\theta,t}\gamma_t + \eta_t) \quad (2.4)$$

where y_t, θ_t are same as in equations (2.1)-(2.2) and $\{\gamma_t\}$ is a sequence of iid positive random variables. $\mu_{y,t}$ and $\mu_{\theta,t}$ are so selected that $E[y_t \mid \boldsymbol{\mathcal{F}_{t-1}}] = 0$ and $E[\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}] = \alpha + \phi(\theta_t - \alpha)$ preserving mean reversion of the returns and the memory effect in volatility respectively. Further ϵ_t and η_t are two sequences of symmetric random variables independent to each other contemporaneously as well as inter-temporally.

The affine combination of positive market factor with symmetric innovation results in a skewed family of distributions. The market factor impact parameters determine the amount and direction of conditional skewness in the corresponding process and hence will be interchangeably called as skewness parameters and impact parameters here onwards. The presence of common market factor in both return and volatility induces the correlation or the feedback effect. The time-dependent impact parameters causes the feedback to be dynamic. It may be remarked here that considering $\lambda_{y,t} = \lambda_y$ and $\lambda_{\theta,t} = \lambda_\theta$, constant feedback model can be obtained. Clearly the volatility asymmetry can now be interpreted in terms of the market factor impacts which has been discussed in detail in subsection 2.3. Henceforth we shall denote the proposed stochastic volatility model with dynamic feedback by SVDF model.

The SVDF model postulated above in equations (2.3)-(2.4) describes a robust class of parametric SV models. Different distributions has been used in SV model to capture the skewness and kurtosis in return. Such models can be obtained as special cases of the proposed structure of the innovations in SVDF model. Some of the significant ones are described as below:

1. Let $\lambda_{y,t} = \lambda_{\theta,t} = 0$, $\epsilon_t \sim N(0,1)$ and $\eta_t \sim N(0,\sigma^2)$ be independent processes to obtain the usual SV model with Gaussian errors (Jacquier et al. 1994)

2. Let $\lambda_{y,t} = \lambda_{\theta,t} = 0$, $\epsilon_t \sim t_\nu$, $\eta_t \sim N(0,\sigma^2)$ and they are independent which leads to the SV model with t-errors (SVt) in return

3. Let $\lambda_{\theta,t} = 0$ and γ_t be standard half-normal variate. Further, let $\epsilon_t \sim N(0,1)$ and independent of $\eta_t \sim N(0,\sigma^2)$ and both ϵ_t and η_t are independent of γ_t which results in the SV model with returns distributed as a variant of Skew-Normal distribution (Tsiotas 2011).

4. Set $\lambda_{\theta,t} = 0$, γ_t as half-t_ν variate. In addition $\epsilon_t \sim t_\nu$ and $\eta_t \sim N(0,\sigma^2)$ and are independent of each other as well as γ_t. This leads to the SV model with Skew-t returns.

5. Let $\lambda_{\theta,t} = 0$ and γ_t be distributed as Generalized Inverse Gaussian distribution. Further, let $\epsilon_t = \sqrt{\gamma_t}\epsilon_t^*$, where ϵ_t^* are $NID(0,1)$ variates independent of γ_t and η_t with

$\eta_t \sim N(0, \sigma^2)$ to obtain the SV model with generalized hyperbolic Skew-t returns (Aas & Haff 2006).

However, in what follows we assume the independence between γ_t, ϵ_t and η_t, $\forall t = 1, 2, \ldots$.

The assumption of dynamic nature of impact parameters $\lambda_{y,t}$ and $\lambda_{\theta,t}$ in the above model immediately results in a serious issue of violating the weak stationarity of the auto-regressive structure of the volatility process as stated in (2.4), which in turn may lead the process to explode as its future variance may increases indefinitely with time lag. To avoid this issue and yet to incorporate the dynamic nature of market factor impacts I first introduce the concept of bounded stationarity in the following subsection and describe some characteristics of θ_t with respect to bounded stationarity.

2.1 Bounded Stationarity For Volatility Process

The bounded stationarity of a discrete time stochastic process is defined as follows.

Bounded Stationarity: Let X_t be a discrete time stochastic process such that its 1^{st} and 2^{nd} moments exist. The process is defined to be bounded stationary if $E[X_t] < M$ and $Cov(X_t, X_{t-k}) < V$; M and V being finite real numbers and k is any integer.

Taking $k = 0$ in the above definition we get the condition $V(y_t) < V$ on the variance for bounded stationarity.

Remark: Notice that, if the 1^{st} and 2^{nd} order moments of a bounded stationary time series are constant, then the series is weak stationary. Hence the weak stationarity is a particular case of bounded stationarity. Further suppose the 1^{st} and 2^{nd} moments of a locally weak stationary series, *viz.* $y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_T}$, $\tau \in I$(an index set), be given by μ_τ and σ^2_τ. If μ_τ and σ^2_τ are finite for all $\tau \in I$, then setting $M = \sup_{\tau \in I} \mu_\tau$ and $V = \sup_{\tau \in I} \sigma_\tau$ we observe that a locally weak stationary series is bounded stationary. Strict stationarity is readily observed to be a special case of bounded stationarity.

Based on the above definition of bounded stationarity, the conditions required for bounded stationarity of the volatility process in (2.4) is derived in the following theorem.

Theorem 2.1 *Consider a sequence of independent positive random variables γ_t and define an auto-regressive process of the form*

$$\theta_t = \alpha + \phi(\theta_{t-1} - \alpha) + a_t, \tag{2.5}$$

where $a_t = \lambda_t \gamma_t + \eta_t$, $\lambda_t \in \mathcal{R}$ and η_t are zero mean and constant variance iid random variables independent of γ_t. Further the sequence, a_t is also assumed to be independent of $\theta_{t'}$, $\forall t' < t$. Assuming that the 1^{st} two moments of η_t exists, the following results hold

1. *$E[\theta_t]$ is finite $\forall\ t$ if $\mid \phi \mid < 1$.*

2. *$V(\theta_t)$ is given by*

$$\Upsilon_t(0) = V(\theta_t) = \sigma^2(1 + \phi^2 + \phi^4 + \ldots) + \delta^2 \sum_{k=1}^{\infty} \phi^{2k} \lambda_{t-k}^2, \tag{2.6}$$

   where $\delta = V(\gamma_t)$. Further $\Upsilon_t(0)$ is non-negative and bounded if $\mid \phi \mid < 1$ and $\mid \lambda_t \mid \leq \lambda$, $\lambda > 0\ \forall t$, in which case

$$\Upsilon_t(0) \leq \frac{\sigma^2 + \delta^2 \lambda^2}{1 - \phi^2} \tag{2.7}$$

3. *The auto-covariance function of lag k is given by $\Upsilon_t(k) = Cov(\theta_{t+k}, \theta_t)$*

$$\Upsilon_t(k) = Cov(\theta_{t+k}, \theta) = \phi^k \Upsilon_t(0) \leq \phi^k \frac{\sigma^2 + \delta^2 \lambda^2}{1 - \phi^2}\ \forall k. \tag{2.8}$$

Proof. Let $V(\eta_t) = \sigma^2$ and observe that $E[a_t] = 0$ and $V(a_t) = \sigma^2 + \lambda_t^2 \delta^2$, $\forall t = 1, 2, \ldots$. The proof of the results are given as below.

1. Notice that ,

$$\begin{aligned} E[\theta_t] &= \alpha(1 - \phi) + \phi E[\theta_{t-1}] \\ &= \alpha(1 - \phi)(1 + \phi + \phi^2 + \phi^3 + \ldots) \end{aligned}$$

   so that $E[\theta_t]$ exists finitely if $\mid \phi \mid < 1$.

2.

$$
\begin{aligned}
\Upsilon_t(0) &= \phi^2 V(\theta_{t-1}) + [\sigma^2 + \delta^2\lambda_t^2] \\
&= \phi^2[\phi^2 V(\theta_{t-2}) + \sigma^2 + \delta^2\lambda_{t-1}^2] + [\sigma^2 + \delta^2\lambda_t^2] \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \\
&= \sigma^2(1 + \phi^2 + \phi^4 + \ldots) + \delta^2[\lambda_t^2 + \phi^2\lambda_{t-1}^2 + \phi^4\lambda_{t-1}^4 \ldots]
\end{aligned}
$$

and if $\mid \phi \mid < 1$ then

$$
= \frac{\sigma^2}{1-\phi^2} + \delta^2 \sum_{k=0}^{\infty} \phi^{2k}\lambda_{t-k}^2.
$$

Further if the condition $\mid \lambda_t \mid \leq \lambda,\ \lambda > 0$ hold for all t, then the bound is immediate from the expression of $\Upsilon_t(0)$.

3. The autocovariance function $\Upsilon_t(k)$ is given by

$$
\begin{aligned}
\Upsilon_t(k) &= E[(\theta_{t+k} - \alpha)(\theta_t - \alpha)] \\
&= \phi E[(\theta_t - \alpha) E[(\theta_{t+k-1} - \alpha) \mid \boldsymbol{\mathcal{F}_{t-1}}]]
\end{aligned}
$$

since a_{t+k} is independent of $\theta_j\ \forall j < t+k$. Thus, by repeated substitutions we get

$$
\Upsilon_t(k) = \phi^k \Upsilon_t(0).
$$

The bound on the auto-covariance function follows from (2.7).

From the above theorem, the following remarks can be made.

Remark: 1. The auto-correlation function is time invariant and depends only on the lag which is similar to the weak stationary time series.

2. The upper bound of the auto-covariance function dampens to zero as the lag increases. Thus, similar to weakly stationary series, the impact of the past realizations decreases with the time horizon. However, unlike the weak stationary series, the auto-covariance of a bounded stationary AR process may not reduce to a time invariant constant with lag in the limit.

3. The k-period ahead forecast for such a series is given by $\hat{\theta}_{t+k} = \alpha + \phi^k(\theta_t - \alpha)$ so that $\lim_{k\to\infty} \hat{\theta}_{t+k} = \alpha$. The forecast error is given by

$$\hat{e}_t(k) = \sum_{j=0}^{k-1} \phi^j a_{t+k-j}$$

.

4. The fore cast error variance is given by:

$$\begin{aligned} V(\hat{e}_t(k)) &= \sum_{j=0}^{k-1} \phi^{2j} V(a_{t+k-j}) \\ &\leq \frac{1-\phi^{2k}}{(1-\phi^2)^2}(\sigma^2 + \delta^2\lambda) \to \frac{\sigma^2 + \delta^2\lambda}{(1-\phi^2)^2}, \text{ as } k \to \infty. \end{aligned}$$

Thus the forecast error variance is bounded above and as the lag increases the sequence of upper bounds also increase. Further, with increasing volatility persistence forecast error variance increases.

The above discussion ensures that although the AR process assumed for volatility process is not weakly stationary but the first two moments of the process are bounded and hence the process and its forecast does not explode with increasing lag. In the following section we discuss on the feedback effect for the proposed SVDF model with general innovation distribution under the assumption of bounded stationarity.

2.2 Dynamic Feedback, Skewness and Leverage In Stochastic Volatility Model

The following lemma provides the expected unconditional return and volatility under the model postulated in (2.3) and (2.4).

Lemma 2.2 *Let y_t and θ_t be the return and volatility at time t and the stochastic volatility model describing the evolution of y_t and θ_t be given as in (2.3-2.4). Suppose ϵ_t is distributed with men 0 and variance unity and is independent of η_t which is distributed with mean 0 and* *variance σ^2. Further suppose that the moment generating functions (MGF) of γ_t (denoted by $M_{\gamma_t}(u)$, $\forall\, u \in \mathcal{R}$) and η_t (denoted by $M_{\eta_t}(u)$, $\forall\, u \in \mathcal{R}$) exist and the first two derivatives of the MGFs are denoted as $M'_X(u)$ and $M''_X(u)$ respectively, $X \in \{\gamma_t, \eta_t\},\ t = 1, 2, \ldots T$. Under the above postulates the following results hold:*

1. $\mu_{y,t} = -A_{t-1}\lambda_{y,t}M_{\eta_t}\left(\frac{1}{2}\right)M'_{\gamma_t}\left(\frac{\lambda_{\theta_t}}{2}\right)$, *where* $A_{t-1} = e^{\frac{\alpha+\phi(\theta_{t-1}-\alpha)+\mu_{\theta,t}}{2}}$

2. $V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) = A^2_{t-1}\left[M_{\eta_t}(1)\left\{\lambda^2_{y,t}M''_{\gamma^*_t}(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t})\right\} - \lambda^2_{y,t}M^2_{\eta_t}\left(\frac{1}{2}\right)M'_{\gamma_t}{}^2\left(\frac{\lambda_{\theta,t}}{2}\right)\right]$

3. $\mu_{\theta,t} = -\lambda_{\theta,t}E(\gamma_t)$

4. $V(\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}) = \lambda^2_{\theta,t}V(\gamma_t) + \sigma^2$

Proof. 1. Define $Z_{y,t} = \lambda_{y,t}\gamma_t + \epsilon_t$ and $Z_{\theta,t} = \lambda_{\theta,t}\gamma_t + \eta_t$ and assume $E[y_t \mid \boldsymbol{\mathcal{F}_{t-1}}] = 0$ to maintain the mean reversibility of the mean corrected returns (y_t). Denoting $e^{\frac{\alpha+\phi(\theta_{t-1}-\alpha)+\mu_{\theta,t}}{2}}$ by A_{t-1}, $\mu_{y,t}$ is given as follows

$$\begin{aligned}\mu_{y,t} &= -A_{t-1}E[e^{\frac{Z_{\theta,t}}{2}}Z_{y,t} \mid \boldsymbol{\mathcal{F}_{t-1}}] \\ &= -A_{t-1}\lambda_{y,t}M_{\eta_t}\left(\frac{1}{2}\right)M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\end{aligned} \tag{2.9}$$

where $M'_{\gamma_t}(u) = \frac{d}{du}M_{\gamma_t}(u)$.

2. Denoting $e^{\frac{\theta_t}{2}}Z_{y,t}$ in (2.3) by R_t, $\forall t = 1, 2, \ldots T$, the return variance is obtained as

$$\begin{aligned}V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) &= E[y_t^2 \mid \boldsymbol{\mathcal{F}_{t-1}}] \\ &= E[R_t^2 \mid \boldsymbol{\mathcal{F}_{t-1}}] - \mu^2_{y,t} \\ &= A^2_{t-1}M_{\eta_t}(1)[\lambda^2_{y,t}M''_{\gamma_t}(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t})] - \mu^2_{y,t} \\ &= A^2_{t-1}\left[M_{\eta_t}(1)\left\{\lambda^2_{y,t}M''_{\gamma_t}(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t})\right\} - \lambda^2_{y,t}M^2_{\eta_t}\left(\frac{1}{2}\right)M'_{\gamma_t}{}^2\left(\frac{\lambda_{\theta,t}}{2}\right)\right]\end{aligned} \tag{2.10}$$

where $M''_{\gamma_t}(u) = \frac{d^2}{du^2}M_{\gamma_t}(u)$.

3. Let $\boldsymbol{\mathcal{F}_{t-1}}$ be the information set available up to time $t-1$. Note that

$$\begin{aligned} E[\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}] &= \alpha + \phi(\theta_{t-1} - \alpha) \\ \Rightarrow \mu_{\theta,t} &= -\lambda_{\theta,t} E[\gamma_t] \end{aligned} \tag{2.11}$$

4. The variance of the volatility process is given as

$$V(\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}) = \lambda_{\theta,t}^2 V(\gamma_t) + \sigma^2 \tag{2.12}$$

Corollary 2.3 *Observing that $M_{\eta_t}(1) \geq M_{\eta_t}^2\left(\frac{1}{2}\right)$, the following lower bound can be obtained from (2.10):*

$$V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) \geq A_{t-1}^2 M_{\eta_t}(1)\left[\lambda_{y,t}^2\left\{M_{\gamma_t}''(\lambda_{\theta,t}) - {M_{\gamma_t}'}^2\left(\frac{\lambda_{\theta,t}}{2}\right)\right\} + M_{\gamma_t}(\lambda_{\theta,t})\right] \tag{2.13}$$

Further, letting $\lambda_{\theta,t} \to 0$ the bound in (2.13) reduces to

$$V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}) \geq e^{\alpha+\phi(\theta_{t-1}-\alpha)} M_{\eta_t}(1)\left[\lambda_{y,t}^2 V(\gamma_t) + 1\right] \tag{2.14}$$

The above corollary may be helpful in determining the minimum risk premium for options based on returns y_t. Next I provide an expression for the dynamic feedback effect for SVDF model

Theorem 2.4 *Under the model and the assumptions postulated in lemma 2.2, the dynamic feedback ρ_t is given by*

$$\rho_t = \frac{\lambda_{y,t}\left[\lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right)\left\{M_{\gamma_t}''\left(\frac{\lambda_{\theta,t}}{2}\right) - M_{\gamma_t}'\left(\frac{\lambda_{\theta,t}}{2}\right)E(\gamma_t)\right\} + E\left(\eta_t e^{\frac{\eta_t}{2}}\right)M_{\gamma_t}'\left(\frac{\lambda_{\theta,t}}{2}\right)\right]}{\sqrt{M_{\eta_t}(1)\left\{\lambda_{y,t}^2 M_{\gamma_t}''(\lambda_{\theta,t}) + M_{\gamma_t}(\lambda_{\theta,t})\right\} - \lambda_{y,t}^2 M_{\eta_t}^2\left(\frac{1}{2}\right){M_{\gamma_t}'}^2\left(\frac{\lambda_{\theta,t}}{2}\right)}\sqrt{\lambda_{\theta,t}^2\delta^2 + \sigma^2}}. \tag{2.15}$$

Proof. The conditional covariance between y_t and θ_t given the information set $\boldsymbol{\mathcal{F}_{t-1}}$ can be derived as follows:

$$\begin{aligned}
Cov_{t-1}(y_t, \theta_t) &= Cov(y_t, \theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}, \Omega) \\
&= E[A_{t-1} e^{\frac{\lambda_{\theta,t}\gamma_t + \eta_t}{2}} (\lambda_{\theta,t}\gamma_t + \eta_t)(\lambda_{y,t}\gamma_t + \epsilon_t)] - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} E_{\gamma_t} E\left[e^{\frac{\lambda_{\theta,t}\gamma_t + \eta_t}{2}} (\lambda_{\theta,t}{\gamma_t}^2 + \gamma_t\eta_t)\right] - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} \left\{\lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right) E_{\gamma_t}\left[{\gamma_t}^2 e^{\frac{\lambda_{\theta,t}\gamma_t}{2}}\right] + E_{\gamma_t}\left[\gamma_t e^{\frac{\lambda_{\theta,t}\gamma_t}{2}}\right] E\left[\eta_t e^{\frac{\eta_t}{2}}\right]\right\} - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} \left[\lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right) M''_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + E\left(\eta_t e^{\frac{\eta_t}{2}}\right) M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\right] - \mu_{y,t}\mu_{\theta,t} \\
&= \lambda_{y,t} A_{t-1} \left[\lambda_{\theta,t} M_{\eta_t}\left(\frac{1}{2}\right) \left\{M''_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) - M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) E(\gamma_t)\right\}\right. \\
&+ \left. E\left(\eta_t e^{\frac{\eta_t}{2}}\right) M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\right]
\end{aligned} \tag{2.16}$$

Notice that existence of MGF of η_t ensures existence of $E\left(\eta_t e^{\frac{\eta_t}{2}}\right)$ and hence the expression for feedback in (2.15) is immediate. ∎

As a consequence of the above theorem the distributions of return and volatility shocks can be characterized as follows.

Corollary 2.5 *The class of distributions that can be considered to model the market factor γ_t and the volatility shock η_t, $\forall t = 1, 2, \ldots T$, are the ones admitting MGF so that the feedback effect and hence the conditional or unconditional return moments exist. However, ϵ_t need not be restricted by such property.*

Thus in the remaining part of this paper I assume that the MGF of γ_t and η_t exists $\forall\, t = 1, 2, \ldots T$. Next I describe the leverage effect under the proposed model.

2.3 Dynamic Leverage in SVDF Model

The presence of conditional leverage effect in the proposed model is reflected through the impact of current return on future volatility (Renault 2009). We prove in the following theorem that the conditional expectation of future volatility depends linearly on the current asset return and the direction of the dependence is determined by the return impact parameter as well as the volatility clustering parameter (ϕ).

Theorem 2.6 *In the model described in (2.3-2.4) along the assumptions described in lemma 2.2, the conditional expectation of future volatility given current return is given as follows:*

$$E[\theta_{t+1} \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}] = C_t + D_t \phi \rho_t y_t \tag{2.17}$$

where ρ_t is the dynamic feedback effect, $u_t = \lambda_{y,t}\gamma_t + \epsilon_t,\ v_t = \lambda_{\theta,t}\gamma_t + \eta_t,\ \omega_{x,t} = \sqrt{Var(x_t)},\ x \in \{u, v\}$ and

$$C_t = \alpha + \phi^2(\theta_{t-1} - \alpha) - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] - A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t\mu_{y,t}M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right)M_{\eta_t}\left(-\frac{1}{2}\right)$$

$$D_t = A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right)M_{\eta_t}\left(-\frac{1}{2}\right) > 0$$

The sign of the conditional leverage is determined by the same of the volatility clustering parameter and the direction of the feedback effect.

Proof. Notice that,

$$\begin{aligned} E[\theta_{t+1} \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}] &= E_{\theta_t}\left[E(\theta_{t+1} \mid \theta_t) \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}\right] \\ &= \alpha + \phi^2(\theta_{t-1} - \alpha) + \phi\mu_{\theta,t} + \phi E[(v_t) \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}], \end{aligned}$$

since γ_{t+1} is independent of y_t and its marginal expectation exists. Let $v_t' = \frac{v_t - E(v_t)}{\omega_{v,t}}$ and $u_t' = \frac{u_t - E(u_t)}{\omega_{u,t}}$ where $\omega_{v,t} = \sqrt{Var(v_t)}$, $\omega_{u,t} = \sqrt{Var(u_t)}$. Further, define $w_t = \frac{v_t' - \rho_t u_t'}{\sqrt{1-\rho_t^2}}$, where ρ_t is the correlation between u_t and v_t. Notice that w_t and u_t' are uncorrelated and $E[w_t] = 0$.

Hence,

$$\begin{aligned}
E[\theta_{t+1} \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}] &= \alpha_{t-1} + \phi\omega_{v,t} E\left[v_t' \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}\right], \\
&\quad (\text{ where } \alpha_{t-1} = \alpha + \phi^2(\theta_{t-1} - \alpha) \text{ and } \mu_{\theta,t} = -E[v_t \mid \boldsymbol{\mathcal{F}_{t-1}}]) \\
&= \alpha_{t-1} - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] + \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E\left[u_t \mid y_t, \boldsymbol{\mathcal{F}_{t-1}}\right] \\
&= \alpha_{t-1} - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] + A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t (y_t - \mu_{y,t}) E\left[e^{-\frac{v_t}{2}} \mid \boldsymbol{\mathcal{F}_{t-1}}\right] \\
&\qquad\qquad (\text{ where } A_{t-1} \text{ is defined above }) \\
&= C_t + A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t y_t M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right) M_{\eta_t}\left(-\frac{1}{2}\right) \\
&= C_t + D_t\phi\rho_t y_t \qquad (2.18)
\end{aligned}$$

where $C_t = \alpha_{t-1} - \frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t E[u_t \mid \boldsymbol{\mathcal{F}_{t-1}}] - A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}}\phi\rho_t\mu_{y,t} M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right) M_{\eta_t}\left(-\frac{1}{2}\right)$ and $D_t = A_{t-1}\frac{\omega_{v,t}}{\omega_{u,t}} M_{\gamma_t}\left(-\frac{\lambda_{\theta,t}}{2}\right) M_{\eta_t}\left(-\frac{1}{2}\right) > 0$. Hence the sign of the dynamic leverage depends on the sign of feedback effect and the volatility clustering parameter. In particular if the feedback effect and the volatility clustering parameter are of opposite sign then the future volatility is negatively correlated to the current return. ∎

2.4 Dynamic Skewness

This subsection attempts to explain the conditional return skewness in terms of the impact parameters. The following theorem provides an expression for the conditional skewness.

Theorem 2.7 *In the model described in (2.3-2.4) along the assumptions described in lemma 2.2, the conditional skewness of return is given as follows:*

$$Sk_t = \frac{\Psi_t^3 + 3\Lambda_t\Psi_t + 3\Delta_t}{\Lambda_t^{\frac{3}{2}}} \qquad (2.19)$$

where $\Psi_t = \frac{\mu_{y,t}}{A_{t-1}}$, $\Lambda_t = \frac{V\left(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}\right)}{A_{t-1}^2}$ *and*

$$\frac{\Delta_t}{A_{t-1}^3} = M_{\eta_t}\left(\frac{3}{2}\right)\left[\lambda_{y,t}^3 M_{\gamma_t}'''\left(\frac{3\lambda_{\theta,t}}{2}\right) + 3\lambda_{y,t} M_{\gamma_t}'\left(\frac{3\lambda_{\theta,t}}{2}\right)\right] \qquad (2.20)$$

$M_X'''(u) = \frac{d^3}{du^3} M_X(u)$.

Proof. Simple algebraic manipulation will show that

$$E\left[y_t^3 \mid \boldsymbol{\mathcal{F}_{t-1}}\right] = 3E\left[e^{\frac{3Z_{\theta,t}}{2}} Z_{y,t}^3 \mid \boldsymbol{\mathcal{F}_{t-1}}\right] + 3\mu_{y,t} V\left(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}\right) + \mu_{y,t}^3$$

where $Z_{\theta,t}$ and $Z_{y,t}$ are defined as in theorem 2.2. Further,

$$\begin{aligned} E\left[e^{\frac{3Z_{\theta,t}}{2}} Z_{y,t}^3 \mid \boldsymbol{\mathcal{F}_{t-1}}\right] &= A_{t-1}^3 E\left[e^{\frac{3Z_{\theta,t}}{2}} \left(\lambda_{y,t}^3 \gamma_t^3 + 3\lambda_{y,t}\gamma_t\epsilon_t^2\right) \mid \boldsymbol{\mathcal{F}_{t-1}}\right] \\ &= M_{\eta_t}\left(\frac{3}{2}\right)\left[\lambda_{y,t}^3 M_{\gamma_t}'''\left(\frac{3\lambda_{\theta,t}}{2}\right) + 3\lambda_{y,t} M_{\gamma_t}'\left(\frac{3\lambda_{\theta,t}}{2}\right)\right] \end{aligned}$$

$\blacksquare$

In the above expression we notice that the conditional skewness is not dependent on the expected volatility or the persistence. Only the impact parameters and the variance of the volatility distribution contributes to the conditional return skewness. Thus the model disentangles the effect of past volatility from the return skewness.

It is difficult to gain further insight on the dynamic leverage effect without assuming particular distributions for γ_t, ϵ_t and η_t. In the following section we make specific assumptions about the distributions of the market factor and return and volatility innovations.

3 Dynamic Leverage In Joint SV Model With Skewness and Kurtosis

The SVDF model proposed above aims to capture the skewness in returns and the dynamic nature of the feedback effect together. In this section we first inspect the SVDF model for skewed returns. In particular, I provide the expression for the feedback effect and conditional skewness and their interpretation in terms of the impact parameters.

3.1 Gaussian SVDF Model

I assume that $\gamma_t \sim HN(0,1)$, $HN(0,1)$ being the standard half-normal distribution and $\epsilon_t \sim N(0,1)$, $\eta_t \sim N(0,\sigma^2)$, $\forall t = 1,2,\ldots$ in addition to the assumptions made in the SVDF model. The expression of feedback effect (ρ_t) can be derived in a similar manner as in theorem 2.4. First we state the following useful lemma related to the moment generating function of standard half normal distribution.

Lemma 3.1 *Let us consider a standard half normal distribution random variable X with mean and let, for any $t \in \mathbf{R}$, $M_X(t)$ be the moment-generating function (MGF) of X. Then,*

$$M'_X(t) = \frac{d}{dt}M_X(t) = tM_X(t) + \sqrt{\frac{2}{\pi}} \tag{3.21}$$

$$M''_X(t) = \frac{d^2}{dt^2}M_X(t) = \{1+t^2\}M_X(t) + t\sqrt{\frac{2}{\pi}} \tag{3.22}$$

Proof. The MGF of X, $M_X(t),\ t \in \mathbf{R}$ is given by

$$\begin{aligned}
M_X(t) &= E[e^{tX}] \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{tx}e^{-\frac{x^2}{2}}dx \\
&= \sqrt{\frac{2}{\pi}}e^{\frac{t^2}{2}}\int_0^\infty e^{-\frac{(x-t)^2}{2}}dx \\
&= \sqrt{\frac{2}{\pi}}e^{\frac{t^2}{2}}\int_{-t}^\infty e^{-\frac{z^2}{2}}dx, \quad \text{letting } z = x - t \\
&= \sqrt{\frac{2}{\pi}}e^{\frac{t^2}{2}}\left[\frac{\sqrt{2\pi}}{2} + \int_{-t}^0 e^{-\frac{z^2}{2}}dz\right] \\
&= e^{\frac{t^2}{2}}\left[1 + \sqrt{\frac{2}{\pi}}\sqrt{2}\int_{-\frac{t}{\sqrt{2}}}^0 e^{-w^2}dw\right], \quad [\text{ letting } z = w\sqrt{2}] \\
&= e^{\frac{t^2}{2}}\left[1 + \frac{2}{\sqrt{\pi}}\int_0^{\frac{t}{\sqrt{2}}} e^{-w^2}dw\right] \\
&= e^{\frac{t^2}{2}}\left[1 + erf\left(\frac{t}{\sqrt{2}}\right)\right], \text{ where } erf(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-w^2}dw \\
&= 2e^{\frac{t^2}{2}}\Phi(t), [\text{ since } erf(u) + 1 = 2\Phi(u\sqrt{2})]
\end{aligned} \tag{3.23}$$

and hence,

$$M'_X(t) = tM_X(t) + \sqrt{\frac{2}{\pi}} = 2te^{\frac{t^2}{2}}\Phi(t) + \sqrt{\frac{2}{\pi}}. \tag{3.24}$$

Further, differentiating (3.21) with respect to t and substituting the expression for $M'_X(t)$ we get

$$M''_X(t) = \{1 + t^2\}M_X(t) + t\sqrt{\frac{2}{\pi}}. \tag{3.25}$$

$\blacksquare$

In the following theorem we derive the expression for dynamic leverage (ρ_t) under the model postulated in (2.3-2.4) and the distributional assumptions stated above and state some sufficient conditions in terms of impact parameters for negative leverage.

Theorem 3.2 *Let y_t and θ_t be the return and volatility at time t and the stochastic volatility model describing the evolution of y_t and θ_t be given as in (2.3-2.4) where γ_t follows standard half normal distribution. Further γ_t is assumed to be independent of the normal variates ϵ_t and η_t which are independent among themselves with mean 0 and variances 1 and σ^2 respectively, $\forall t = 1, 2, \dots n$. Under this model the dynamic leverage effect ρ_t is given by*

$$\rho_t = \frac{\lambda_{y,t}\left[\lambda_{\theta,t}M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\left\{\frac{\lambda^2_{\theta,t}}{4} + 1 + \frac{\sigma^2}{4} - \frac{\lambda_{\theta,t}}{\sqrt{2\pi}}\right\} + \frac{1}{\sqrt{2\pi}}\left(\sigma^2 + \lambda^2_{\theta,t}\right) - \frac{2\lambda_{\theta,t}}{\pi}\right]}{\sqrt{e^{\frac{\sigma^2}{4}}\left\{M_{\gamma_t}(\lambda_{\theta,t})[\lambda^2_{y,t}(\lambda^2_{\theta,t}+1)+1] + \sqrt{\frac{2}{\pi}}\lambda_{\theta,t}\lambda^2_{y,t}\right\} - \lambda^2_{y,t}{M'_{\gamma_t}}^2\left(\frac{\lambda_{\theta,t}}{2}\right)}\sqrt{\lambda^2_{\theta,t}\left(1-\frac{2}{\pi}\right)+\sigma^2}} \tag{3.26}$$

where $M'_{\gamma_t}(u)$ and $M''_{\gamma_t}(u)$ are as defined in (3.21-3.22).

Proof. Notice that, here $M_{\eta_t}\left(\frac{1}{2}\right) = e^{\frac{\sigma^2}{8}}$ and hence from (2.9)

$$\mu_{y,t} = -A_{t-1}\lambda_{y,t}e^{\frac{\sigma^2}{8}}M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) \tag{3.27}$$

Further, observing that $M_{\eta_t}(1) = e^{\frac{\sigma^2}{2}}$, expressions in (2.12) and (2.10) leads to

$$V(\theta_t \mid \boldsymbol{\mathcal{F}_{t-1}}, \Omega) = \lambda^2_{\theta,t}\left(1 - \frac{2}{\pi}\right) + \sigma^2 \tag{3.28}$$

and

$$
\begin{aligned}
V(y_t \mid \boldsymbol{\mathcal{F}_{t-1}}, \Omega) &= A_{t-1}^2 e^{\frac{\sigma^2}{2}} \left\{ M_{\gamma_t}(\lambda_{\theta,t})[\lambda_{y,t}^2(\lambda_{\theta,t}^2+1)+1] + \sqrt{\frac{2}{\pi}}\lambda_{\theta,t}\lambda_{y,t}^2 \right\} - \mu_{y,t}^2 \\
&= A_{t-1}^2 e^{\frac{\sigma^2}{4}} \left[e^{\frac{\sigma^2}{4}} \left\{ M_{\gamma_t}(\lambda_{\theta,t})[\lambda_{y,t}^2(\lambda_{\theta,t}^2+1)+1] + \sqrt{\frac{2}{\pi}}\lambda_{\theta,t}\lambda_{y,t}^2 \right\} \right. \\
&\quad \left. -\lambda_{y,t}^2 {M'_{\gamma_t}}^2 \left(\frac{\lambda_{\theta,t}}{2}\right) \right]. \qquad (3.29)
\end{aligned}
$$

Further, $E\left[\eta_t e^{\frac{\eta_t}{2}}\right] = \frac{\sigma^2}{2} e^{\frac{\sigma^2}{8}}$ and hence from (2.16) :

$$
\begin{aligned}
Cov_{t-1}(y_t, \theta_t) &= A_{t-1} e^{\frac{\sigma^2}{8}} \left[\lambda_{\theta,t}\lambda_{y,t} M''_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + \lambda_{y,t}\frac{\sigma^2}{2} M'_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) \right] - \mu_{y,t}\mu_{\theta,t} \\
&= A_{t-1} e^{\frac{\sigma^2}{8}} \left[\lambda_{y,t}\lambda_{\theta,t} \left\{ \left(\frac{\lambda_{\theta,t}^2}{4}+1\right) M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + \sqrt{\frac{2}{\pi}}\frac{\lambda_{\theta,t}}{2} \right\} \right. \\
&+ \left. \lambda_{y,t}\frac{\sigma^2}{2} \left\{ \frac{\lambda_{\theta,t}}{2} M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) + \sqrt{\frac{2}{\pi}} \right\} \right] - \mu_{y,t}\mu_{\theta,t} \\
&= A_{t-1} e^{\frac{\sigma^2}{8}} \lambda_{y,t} \left[\lambda_{\theta,t} M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right) \left\{ \frac{\lambda_{\theta,t}^2}{4} + 1 + \frac{\sigma^2}{4} - \frac{\lambda_{\theta,t}}{\sqrt{2\pi}} \right\} \right. \\
&+ \left. \frac{1}{\sqrt{2\pi}}\left(\sigma^2+\lambda_{\theta,t}^2\right) - \frac{2\lambda_{\theta,t}}{\pi} \right] \qquad (3.30)
\end{aligned}
$$

Hence the expression for leverage in (3.26) is immediate. ∎

Remark: The correlation coefficient varies with respect the impact parameters as well as the variance of the volatility. A necessary and sufficient condition for the feedback effect to be negative is that $\lambda_{y,t}$ and $\kappa_t = \lambda_{\theta,t} M_{\gamma_t}\left(\frac{\lambda_{\theta,t}}{2}\right)\left\{\frac{\lambda_{\theta,t}^2}{4} + 1 + \frac{\sigma^2}{4} - \frac{\lambda_{\theta,t}}{\sqrt{2\pi}}\right\} + \frac{1}{\sqrt{2\pi}}\left(\sigma^2 + \lambda_{\theta,t}^2\right) - \frac{2\lambda_{\theta,t}}{\pi}$ are of opposite sign ($\forall \sigma > 0$). Notice that κ_t has a minimum at $\lambda_{\theta,t}^{min}$ for each $\sigma > 0$ with minimum value κ_t^{min} at each time point $t = 1, 2, \ldots, T$. Figure 1 in Appendix A plots κ_t^{min} against σ for any t.

As evident from figure 1, κ_t^{min} exceeds zero and numerical computation shows that the corresponding $\sigma = 0.74182$. Thus κ_t can take negative values only for $\sigma \in (0, 0.7419819)\{= I_{min}\}$. Thus, to find the range of $\lambda_{\theta,t}$ so that $\kappa_t < 0$, we restrict σ within this interval. Further, numerically it can be verified that there are only two roots to $\kappa_t = 0$, say $\lambda_{\theta,t}^1 < \lambda_{\theta,t}^2$, for $\sigma \in I_{min}$. Figure 2 in Appendix A show the plots of $\lambda_{\theta,t}^1$ and $\lambda_{\theta,t}^2$ against $\sigma \in I_{min}$.

It is clear from the above figures that the interval within which $\kappa_t < 0$ reduces with increasing σ. Thus necessary and sufficient condition for feedback to be negative is translates to either $\lambda_{y,t} < 0$ and $\lambda_{\theta,t}$ lies out side the interval $(\lambda^1_{\theta,t}, \lambda^2_{\theta,t})$ or the other way around where the limits $\lambda^i_{\theta,t},\ i = 1, 2$ depend on the variance of the volatility process.

Remark: Figures (3-4) given in the appendix provide the feedback effect surface corresponding to impact parameters for given volatility variances.

It may be noticed from the above figures that as $\sigma \to \infty$, the impact surface closes to the constant plane at zero. Observing that very high volatility variance induces positive probability for the event that realization of conditional volatility is far away from its conditional mean. Such a case may happen in times of bubbles and crashes. One possible explanation for almost zero feedback could be that during such time, market factor impacts are outperformed by the random shocks and hence feedback appears insignificant. In the particular case of no impact of market factor on volatility ($\lambda_{\theta,t} \to 0$), simple algebraic calculation will reveal that $\rho_t \to \frac{\lambda_{y,t}\sigma}{\sqrt{2\pi e^{\frac{\sigma^2}{4}}(\lambda^2_{y,t}+1)-2\lambda^2_{y,t}}}$, which tends to 0 with increasing σ.

Remark: Notice that for a standard half-normal random variable X,

$$M'''_X(u) = \frac{d^3}{du^3}M_X(u) = u(u^2+3)M_X(u) + \sqrt{\frac{2}{\pi}}(u+1) \tag{3.31}$$

Hence, from theorem 2.7, the conditional skewness could be derived.

4 Discussion

In this paper the inter connection between feedback effect and return skewness has been established which lead to further interesting insights. First, I have developed a parsimonious single factor SV model to explain the linkage between return skewness and feedback effect as mentioned by Renault (2009). This model leads to a simple characterization of the admissible distributions for return and its volatility. Precisely, the skewness of returns has been shown as a perturbation of symmetric return error with a positive "market factor" common to both return and volatility and the feedback is generated as a result of the shared factor between return and volatility. Secondly, the reaction of the feedback effect to the variance of the volatility process has been shown. The interesting fact that could be noticed from the feedback surface is that if the volatility process itself has very high variance then the market impacts matter in infinitesimal. The third issue that has been addressed here is to accommodate the dynamic nature of the skewness as mentioned in Boyer et al. (2010). In particular, the concept of bounded stationarity has been introduced as a generalization of weakly stationary process and the non-stationarity arising out of dynamic skewness has been tackled with the bounded stationarity which enables finite forecasts of volatility process.

This paper also leaves scope for further research both in theoretical and application aspects of SV model. The heavy tail nature of returns could be generated using a scaling to the skew family of error distributions. Since the model is saturated, it is difficult to estimate the parameters in a frequentist set up. As a solution Bayesian Monte Carlo Markov Chain methods could be used to resolve the parameter estimation problem (Jacquier et al. 2004). Further, in case of multivariate SV models prior information on feedback could be used to elicit admissible prior distribution for MCMC estimation. On the application part I plan to apply the model on S&P500 returns and volatility index of Chicago board of options exchange and compare in terms of Bayesian model complexity measures.

A

Appendix

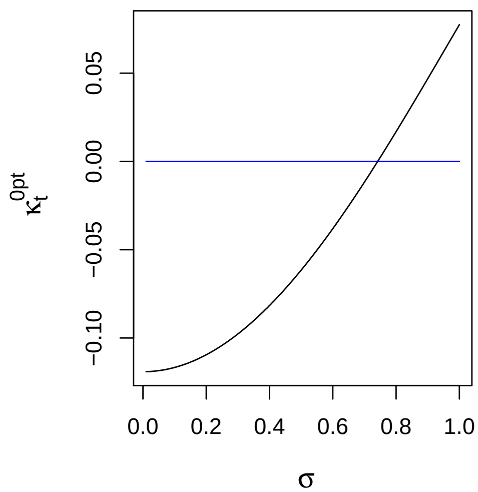

Figure 1: Minimum value of κ_t for different values of σ

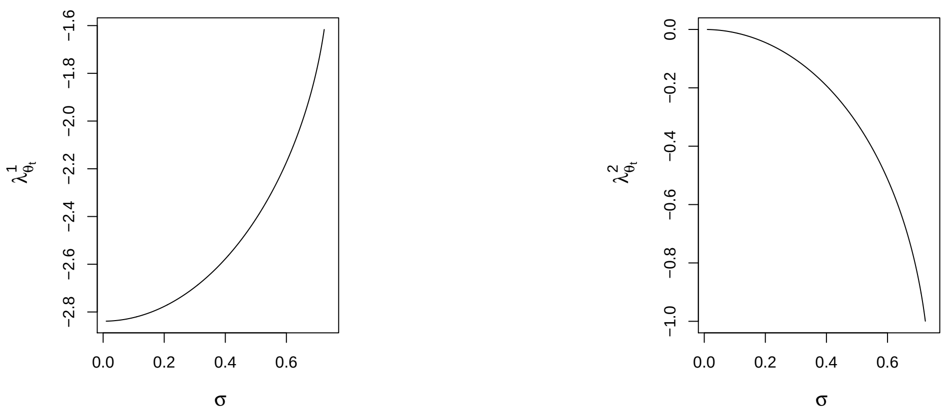

Figure 2: Plot of $\lambda^1_{\theta,t}$ and $\lambda^2_{\theta,t}$

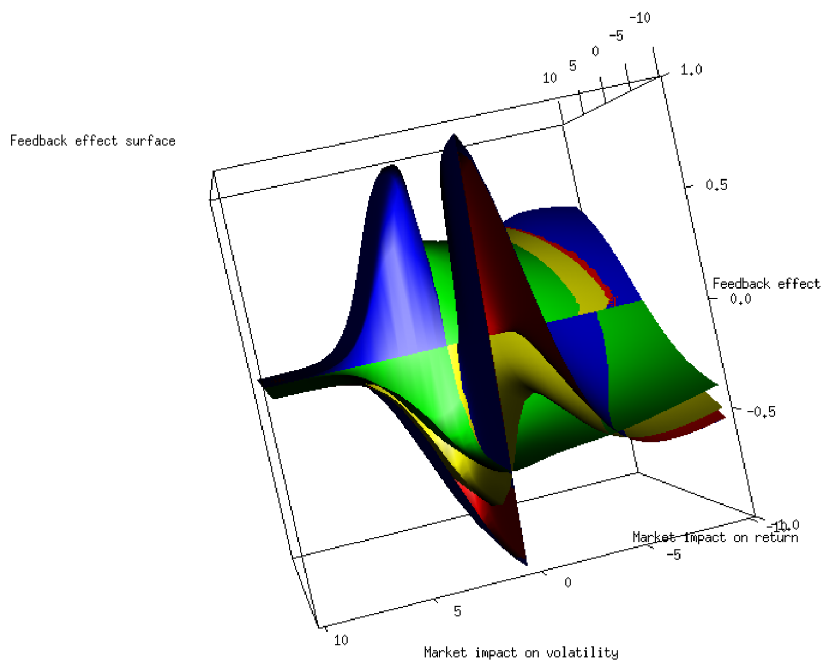

Figure 3: Feedback surface plots for σ =0.001(blue), 0.1(yellow), 1(red), 2(green)

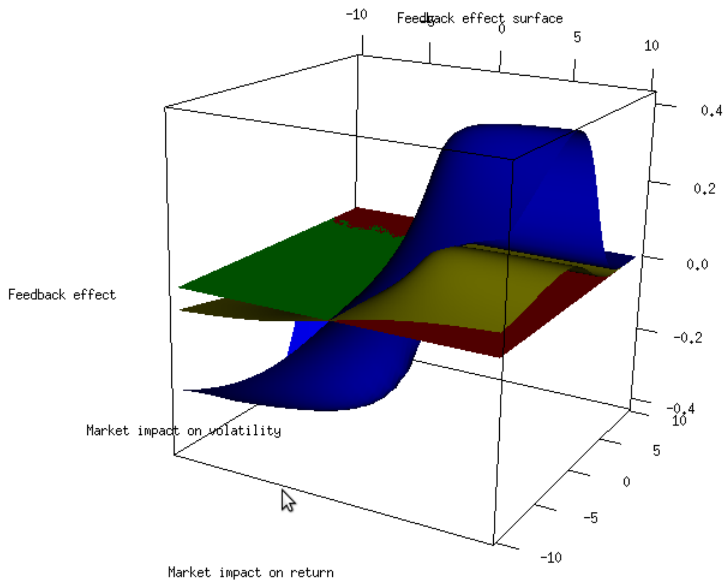

Figure 4: Feedback surface plots for σ =3 (blue), 5 (yellow), 10 (red), 20 (green)

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