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Ordering properties of order statistics from heterogeneous exponentiated Weibull models

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ABSTRACT

In this paper we stochastically compare two parallel systems each having heterogeneous exponentiated Weibull components. These comparisons are made with respect to reversed hazard rate ordering and likelihood ratio ordering. Similar comparisons are also made for two systems with component lives following multiple outlier exponentiated Weibull model.

1. Introduction

Order statistics have a remarkable contribution in both theory and practice. It has a prominent role in statistics, applied probability, reliability theory, operations research, actuarial science, auction theory, hydrology and many other related and unrelated areas. Parallel and series systems are the building blocks of many complex coherent systems in reliability theory. If $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics corresponding to the random variables $X_1, X_2, \ldots, X_n$, then the lifetime of a series system corresponds to the smallest order statistic $X_{1:n}$ and that of a parallel system is represented by the largest order statistic $X_{n:n}$. Although different properties of order statistics from homogeneous populations have been studied in detail in the literature, not much work is available for order statistics from heterogeneous populations, due to its complicated nature of expressions. For properties of order statistics for independent and non-identically distributed random variables, one may refer to David and Nagaraja (2003). Stochastic comparisons of parallel systems of heterogeneous components with exponential, gamma, Weibull and generalized exponential (GE) lifetimes have been studied by several authors. One may refer to Dykstra et al. (1997), Misra and Misra (2013), Zhao and Balakrishnan (2011, 2012), Torrado and Kochar (2015), Kundu et al. (2016) and the references there in.

The three-parameter exponentiated Weibull (EW) distribution was proposed by Mudholkar and Srivastava (1993) by exponentiating the two-parameter Weibull distribution and was later analyzed extensively by Mudholkar et al. (1995, 1996) and Mudholkar and Hutson (1996). The EW distribution is shown to model non-monotone hazard rates, including the bathtub-shaped hazard rate, and is shown to fit many real life situations well, compared to the conventional exponential,

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gamma or Weibull models. A random variable X is said to have EW (also known as generalized Weibull) distribution with parameters (α, β, λ), written as EW(α, β, λ), if the distribution function of X is given by

$$F(x) = \left(1 - e^{-(\lambda x)^{\beta}}\right)^{\alpha}, \quad x > 0,\ \lambda > 0,\ \beta > 0,\ \alpha > 0,$$

where α and β are the shape parameters and λ is the scale parameter. Recently, Fang and Zhang (2015) have nicely compared two parallel systems of EW components in terms of usual stochastic order, dispersive order and likelihood ratio order. In this paper our main aim is to compare two parallel systems in terms of reversed hazard rate order and likelihood ratio order by comparing the parameters λ and α when the components are from two heterogeneous EW distributions as well as from the multiple outlier EW distributions. These results generalize similar results for GE distribution (taking $\beta = 1$), generalized Rayleigh distribution (taking $\beta = 2$), Weibull distribution (taking $\alpha = 1$) and exponential distribution (taking $\beta = 1$ and $\alpha = 1$).

The organization of the paper is as follows. In Section 2, we have given the required definitions and some useful lemmas which have been used throughout the paper. In Section 3, results related to reversed hazard rate ordering between two order statistics $X_{n:n}$ and $Y_{n:n}$ are given under weak majorization of the parameters α and λ. We have also shown that there exists likelihood ratio ordering between $X_{2:2}$ and $Y_{2:2}$ under certain arrangements of the parameters. For the case of multiple-outlier EW model, the likelihood ratio ordering between $X_{n:n}$ and $Y_{n:n}$ is also established. Finally, Section 4 concludes the paper.

Throughout the paper, the word increasing (resp. decreasing) and nondecreasing (resp. nonincreasing) are used interchangeably, and $\Re$ denotes the set of real numbers $\{x : -\infty < x < \infty\}$. We also write $a \overset{sign}{=} b$ to mean that a and b have the same sign. For any differentiable function $k(\cdot)$, we write $k'(t)$ to denote the first derivative of $k(t)$ with respect to t. The random variables considered in this paper are all nonnegative. By independence of random variables we mean that they are statistically independent.

2. Preliminaries

For two absolutely continuous random variables X and Y with distribution functions $F(\cdot)$ and $G(\cdot)$, density functions $f(\cdot)$ and $g(\cdot)$ and reversed hazard rate functions $r(\cdot)$ and $s(\cdot)$ respectively, X is said to be smaller than Y in (i) *likelihood ratio order* (denoted as $X \le_{lr} Y$), if $\frac{g(t)}{f(t)}$ increases in t, and (ii) *reversed hazard rate order* (denoted as $X \le_{rhr} Y$), if $\frac{G(t)}{F(t)}$ increases in t or equivalently $r(t) \le s(t)$ for all t. For more on different stochastic orders, see Shaked and Shanthikumar (2007).

The notion of majorization (Marshall et al., 2011) is essential for the understanding of the stochastic inequalities for comparing order statistics. Let I^n be an n-dimensional Euclidean space where $I \subseteq \Re$. Further, for any two real vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$, write $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$ and $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$ as the increasing arrangements of the components of the vectors $\mathbf{x}$ and $\mathbf{y}$ respectively. The following definitions may be found in Marshall et al. (2011).

Definition 2.1. (i) The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}, \quad j = 1,\ 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

(ii) The vector $\mathbf{x}$ is said to weakly supermajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{\mathrm{w}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}, \quad \text{for } j = 1, 2, \ldots, n.$$

(iii) The vector $\mathbf{x}$ is said to weakly submajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \succeq_{\mathrm{w}} \mathbf{y}$) if

$$\sum_{i=j}^{n} x_{(i)} \ge \sum_{i=j}^{n} y_{(i)}, \quad \text{for } j = 1, 2, \ldots, n.$$

Definition 2.2. A function $\psi : I^n \to \Re$ is said to be Schur-convex (resp. Schur-concave) on I^n if $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$ implies $\psi(\mathbf{x}) \ge$ (resp. $\le$) $\psi(\mathbf{y})$ *for all* $\mathbf{x},\ \mathbf{y} \in I^n$. □

We next present two useful lemmas which will be used in the next section to prove our main results.

Lemma 2.1. *For* $x, \beta > 0$, $\phi(y) = y\left(e^{yx^{\beta}} - 1\right)^{-1}$ *is decreasing in* $y > 0$. □

The following lemma can be found in Marshall et al. (2011, p. 87) where the parenthetical statements are not given.

Lemma 2.2. *Let* $\varphi : I^n \to \Re$. *Then*

$(a_1, a_2, \ldots, a_n) \succeq_w (b_1, b_2, \ldots, b_n)$ *implies* $\varphi(a_1, a_2, \ldots, a_n) \ge$ *(resp.* $\le$*)* $\varphi(b_1, b_2, \ldots, b_n)$

if, and only if, φ *is increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on* I^n. *Similarly,*

$$(a_1, a_2, \ldots, a_n) \overset{w}{\succeq} (b_1, b_2, \ldots, b_n) \text{ implies } \varphi(a_1, a_2, \ldots, a_n) \geq (\text{resp. } \leq)\ \varphi(b_1, b_2, \ldots, b_n)$$

if, and only if, φ *is decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on* I^n.

3. Main results

Notation 3.1. *Let us first introduce the following notations which will be used in all the upcoming theorems.*

(i) $\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$.
(ii) $\mathcal{E}_+ = \{(x_1, x_2, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \cdots \leq x_n\}$.

For $i = 1, 2, \ldots, n$, let $X_i \sim EW\,(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW\,(\theta_i, \beta, \delta_i)$ be two sets of n independent random variables. If $F_{n:n}\,(\cdot)$ and $G_{n:n}\,(\cdot)$ are the distribution functions of $X_{n:n}$ and $Y_{n:n}$ respectively, then

$$F_{n:n}\,(x) = \prod_{i=1}^{n} \left(1 - e^{-(\lambda_i x)^\beta}\right)^{\alpha_i},$$

and

$$G_{n:n}\,(x) = \prod_{i=1}^{n} \left(1 - e^{-(\delta_i x)^\beta}\right)^{\theta_i}.$$

Again, if $r_{n:n}(\cdot)$ and $s_{n:n}(\cdot)$ are the reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$ respectively then,

$$r_{n:n}\,(x) = \sum_{i=1}^{n} \frac{\alpha_i \beta \lambda_i^\beta x^{\beta-1}}{e^{(\lambda_i x)^\beta} - 1}, \tag{3.1}$$

$$s_{n:n}\,(x) = \sum_{i=1}^{n} \frac{\theta_i \beta \delta_i^\beta x^{\beta-1}}{e^{(\delta_i x)^\beta} - 1}. \tag{3.2}$$

The following two theorems show that under certain conditions on parameters, there exists reversed hazard rate ordering between $X_{n:n}$ and $Y_{n:n}$. Write $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$.

Theorem 3.1. *For* $i = 1, 2, \ldots, n$, *let* X_i *and* Y_i *be two sets of mutually independent random variables with* $X_i \sim EW\,(\alpha_i, \beta, \lambda_i)$ *and* $Y_i \sim EW\,(\alpha_i, \beta, \delta_i)$. *Further, suppose that* $\boldsymbol{\alpha} \in \mathcal{E}_+$ *and* $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$. *Then, for all* $\beta > 0$,

$$\boldsymbol{\lambda} \overset{w}{\succeq} \boldsymbol{\delta} \text{ implies } X_{n:n} \geq_{rhr} Y_{n:n}.$$

Proof. Let $g_i(y) = \frac{\alpha_i y \beta x^{\beta-1}}{e^{yx^\beta} - 1}$. Differentiating $g_i(y)$ with respect to y, we get

$$g_i'(y) = \alpha_i \psi(y),$$

where, by Lemma 2.4 of Fang and Zhang (2015),

$$\psi(y) = \beta x^{\beta-1} \frac{e^{yx^\beta} - 1 - yx^\beta e^{yx^\beta}}{\left(e^{yx^\beta} - 1\right)^2}$$

is increasing in $y \geq 0$, for all $\beta \geq 0$. So, for any two real numbers $a \geq b$, $\psi(a) \geq \psi(b)$. Again, from Lemma 2.1 it follows that $\psi(y) < 0$ for all $y > 0$. Now, if $\boldsymbol{\alpha} \in \mathcal{E}_+$, then $-\alpha_i \psi(a) \leq -\alpha_{i+1} \psi(b)$, which implies that $g_i'(a) \geq g_{i+1}'(b)$. So, by Proposition H.2 of Marshall et al. (2011), $r_{n:n}\,(x)$ is Schur convex. Thus, the result follows from Lemmas 2.1 and 2.2. $\square$

The counterexample given below shows that without the ascending order of the components of the scale parameters and the descending order of the components of the shape parameters Theorem 3.1 may not hold, even if the weak majorization between $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ is replaced by majorization order.

Counterexample 3.1. Let $X_i \sim EW\,(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW\,(\alpha_i, \beta, \delta_i)$, $i = 1, 2, 3$. Now, if $(\lambda_1, \lambda_2, \lambda_3) = (5, 4, 1) \in \mathcal{D}_+$, $(\delta_1, \delta_2, \delta_3) = (4, 4, 2) \in \mathcal{D}_+$, $(\alpha_1, \alpha_2, \alpha_3) = (2, 50, 100) \in \mathcal{E}_+$ and $\beta = 2$ are taken, then from Fig. 2.1 it can be concluded that $X_{3:3} \geq_{rhr} Y_{3:3}$, while $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$. But if we reverse the order of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ then, for same $\boldsymbol{\alpha}$ and β, $r_{3:3}(0.8) - s_{3:3}(0.8) = 2.40613$ and $r_{3:3}(0.5) - s_{3:3}(0.5) = -22.6305$ giving $X_{3:3} \not\geq_{rhr} Y_{3:3}$. It is to be mentioned here that while plotting the curve, the substitution $x = -\ln y$ has been used so that $r_{3:3}(-\ln y) - s_{3:3}(-\ln y) = a(y)$, say.

Theorem 3.1 shows the ordering between $X_{n:n}$ and $Y_{n:n}$ when $\boldsymbol{\lambda}$ majorizes $\boldsymbol{\delta}$ keeping the other parameters same. Now the question arises—what will happen if $\boldsymbol{\alpha}$ majorizes $\boldsymbol{\theta}$ while the scale parameters $\boldsymbol{\lambda}$, and $\boldsymbol{\delta}$ are equal? The theorem given below answers this question.

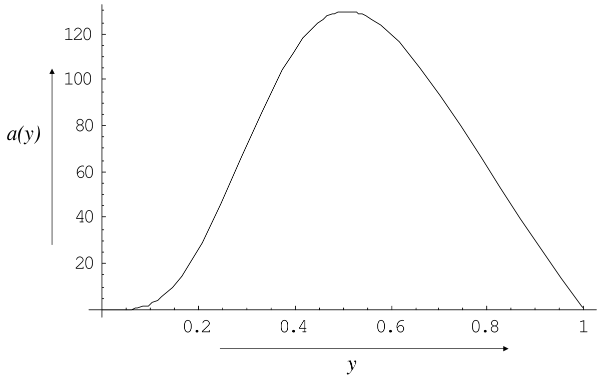

Fig. 2.1. Graph of $a(y)$.

Theorem 3.2. *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EW(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW(\theta_i, \beta, \lambda_i)$. If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\theta}$ and*

(i) $\boldsymbol{\alpha}, \boldsymbol{\theta} \in \mathcal{D}_+$ *and* $\boldsymbol{\lambda} \in \mathcal{E}_+$, *then* $X_{n:n} \geq_{rhr} Y_{n:n}$;

(ii) $\boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_+$, *then* $X_{n:n} \leq_{rhr} Y_{n:n}$.

Proof. Assuming $g(x_i) = x_i$, $u_i = \frac{\lambda_i^\beta \beta x^{\beta-1}}{e^{(\lambda_i x)^\beta}-1}$ and

$$\varphi(\boldsymbol{\alpha}) = \sum_{i=1}^n u_i g(\alpha_i),$$

and by noting the fact that $u_i \in \mathcal{D}_+$ ($u_i \in \mathcal{E}_+$) whenever $\lambda_i \in \mathcal{E}_+$ ($\lambda_i \in \mathcal{D}_+$) (by Lemma 2.1) and using Theorem 3.1 and Theorem 3.2 of Kundu et al. (2016) it can be proved that $r_{n:n}(x)$ is Schur convex under condition (i), and Schur concave under condition (ii). This proves the result. □

The following theorem follows from Lemma 2.2 and Theorem 3.2.

Theorem 3.3. *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EW(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW(\theta_i, \beta, \lambda_i)$.*

(i) *If* $\boldsymbol{\alpha}, \boldsymbol{\theta} \in \mathcal{D}_+$ *and* $\boldsymbol{\lambda} \in \mathcal{E}_+$, *then* $\boldsymbol{\alpha} \succeq_w \boldsymbol{\theta}$ *implies* $X_{n:n} \geq_{rhr} Y_{n:n}$.

(ii) *If* $\boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_+$, *then* $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\theta}$ *implies* $X_{n:n} \leq_{rhr} Y_{n:n}$.

The following theorem shows that, based on the majorization order of the parameters, the ratio of the reversed hazard rate functions of $X_{2:2}$ and $Y_{2:2}$ is monotone.

Theorem 3.4. *For $i = 1, 2$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EW(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW(\alpha_i, \beta, \delta_i)$. If $\boldsymbol{\alpha} \in \mathcal{E}_+$, $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$ and $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$, then, for $0 < \beta \leq 1$,*

$$\frac{r_{2:2}(x)}{s_{2:2}(x)} \text{ is increasing in } x.$$

Proof. Let $u(x) = \frac{x^\beta}{e^{x^\beta}-1}$. So,

$$\frac{r_{2:2}(x)}{s_{2:2}(x)} = \frac{\frac{\alpha_1\beta\lambda_1^\beta x^{\beta-1}}{e^{(\lambda_1 x)^\beta}-1} + \frac{\alpha_2\beta\lambda_2^\beta x^{\beta-1}}{e^{(\lambda_2 x)^\beta}-1}}{\frac{\alpha_1\beta\delta_1^\beta x^{\beta-1}}{e^{(\delta_1 x)^\beta}-1} + \frac{\alpha_2\beta\delta_2^\beta x^{\beta-1}}{e^{(\delta_2 x)^\beta}-1}} = \frac{\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)}{\alpha_1 u(\delta_1 x) + \alpha_2 u(\delta_2 x)} = \eta(x)(say).$$

Now, $u(x)$, by Lemma 3.1 of Torrado and Kochar (2015), is decreasing and convex in $x \geq 0$, for all $0 < \beta \leq 1$. Again, differentiating $u(x)$ with respect to x, we get

$$u'(x) = \frac{\beta}{x} u(x) v(x),$$

where

$$v(x) = 1 - \frac{x^\beta}{1 - e^{-x^\beta}}.$$

By Lemma 3.2 of Torrado and Kochar (2015), $v(x)$ is decreasing in x. Again, differentiating $\eta(x)$ with respect to x, we get

$$\eta'(x) \overset{sign}{=} \left(\alpha_1\lambda_1 u'(\lambda_1 x) + \alpha_2\lambda_2 u'(\lambda_2 x)\right)(\alpha_1 u(\delta_1 x) + \alpha_2 u(\delta_2 x)) - \left(\alpha_1\delta_1 u'(\delta_1 x) + \alpha_2\delta_2 u'(\delta_2 x)\right)(\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)).$$

Thus, it is clear that $\eta(x)$ is increasing in $x \geq 0$ if

$$\Psi(\lambda_1, \lambda_2) = \frac{\alpha_1\lambda_1 u'(\lambda_1 x) + \alpha_2\lambda_2 u'(\lambda_2 x)}{\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)} = \frac{\beta}{x}\frac{\alpha_1 u(\lambda_1 x)v(\lambda_1 x) + \alpha_2 u(\lambda_2 x)v(\lambda_2 x)}{\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)}$$

is Schur-convex in (λ_1, λ_2). Writing $w(x) = u(x)v'(x)(\leq 0)$, we get

$$\frac{\partial\Psi}{\partial\lambda_1} - \frac{\partial\Psi}{\partial\lambda_2} \overset{sign}{=} \left[(\alpha_1\alpha_2(v(\lambda_1 x) - v(\lambda_2 x)))\left(u'(\lambda_1 x)u(\lambda_2 x) + u'(\lambda_2 x)u(\lambda_1 x)\right) + ((\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)))(\alpha_1 w(\lambda_1 x) - \alpha_2 w(\lambda_2 x))\right]. \tag{3.3}$$

Again, by Lemma 3.4 of Torrado and Kochar (2015),

$$w(x) = \frac{\beta e^{2\beta-1}e^{x^\beta}\left(x^\beta + 1 - e^{x^\beta}\right)}{\left(e^{x^\beta} - 1\right)^3}$$

is increasing in $x \geq 0$ for all $0 < \beta \leq 1$.

Now, as $\boldsymbol{\lambda} \in \mathcal{D}_+$ and $v(x)$ is decreasing in x, it can be written that $v(\lambda_1 x) - v(\lambda_2 x) \leq 0$ for all $\lambda_1 \geq \lambda_2$. Thus, by noting the fact that $u(x)$ is decreasing in x, it can be shown that the first term of (3.3) is nonnegative. Again, for $\boldsymbol{\alpha} \in \mathcal{E}_+$, $\alpha_1 w(\lambda_1 x) \geq \alpha_2 w(\lambda_2 x)$. Hence, for all $x \geq 0$, the second term of (3.3) is also nonnegative. Thus, by Lemma 3.1 of Kundu et al. (2016), Ψ is Schur convex in $\boldsymbol{\lambda}$. Thus, $\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta}$ gives $\eta'(x) \geq 0$, proving the result. □

Kundu et al. (2016) have shown that, for generalized exponential distribution, although there does not exist lr ordering between $X_{3:3}$ and $Y_{3:3}$ when $\boldsymbol{\delta}$ is majorized by $\boldsymbol{\lambda}$, the result is true for $n = 2$. Corollary 3.1 shows that the same can be concluded for EW distribution.

Corollary 3.1. *For $i = 1, 2$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EW(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW(\alpha_i, \beta, \delta_i)$. If $\boldsymbol{\alpha} \in \mathcal{E}_+$, $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$ and $0 < \beta \leq 1$, then*

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\delta} \text{ implies } X_{2:2} \geq_{lr} Y_{2:2}.$$

Proof. If $f_{2:2}(\cdot)$ and $g_{2:2}(\cdot)$ denote the density functions of $X_{2:2}$ and $Y_{2:2}$ respectively, then

$$\frac{f_{2:2}(x)}{g_{2:2}(x)} = \frac{F_{2:2}(x)}{G_{2:2}(x)}\cdot\frac{r_{2:2}(x)}{s_{2:2}(x)} \text{ is increasing in } x.$$

This is because the first ratio is increasing in x by Theorem 3.1 and the second ratio is increasing in x by Theorem 3.4. □

The next theorem shows that, if $\boldsymbol{\alpha}$ majorizes $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ is kept fixed, then depending upon certain conditions on $\boldsymbol{\alpha}$, $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, $\frac{r_{2:2}(x)}{s_{2:2}(x)}$ will be monotone.

Theorem 3.5. *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EW(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW(\theta_i, \beta, \lambda_i)$. Further, suppose that $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\theta}$. Then*

(i) $\frac{r_{2:2}(x)}{s_{2:2}(x)}$ *is increasing in x if $\boldsymbol{\alpha}, \boldsymbol{\theta} \in \mathcal{D}_+$ and $\boldsymbol{\lambda} \in \mathcal{E}_+$.*

(ii) $\frac{r_{2:2}(x)}{s_{2:2}(x)}$ *is decreasing in x if $\boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_+$.*

Proof. Following Theorem 3.4, we have

$$\frac{r_{2:2}(x)}{s_{2:2}(x)} = \frac{\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)}{\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x)} = \eta_1(x)(say),$$

where $u(x)$ is as defined in Theorem 3.4. Differentiating $\eta_1(x)$ with respect to x, we get

$$\eta_1'(x) \overset{sign}{=} \left(\alpha_1\lambda_1 u'(\lambda_1 x) + \alpha_2\lambda_2 u'(\lambda_2 x)\right)(\theta_1 u(\lambda_1 x) + \theta_2 u(\lambda_2 x)) - \left(\theta_1\lambda_1 u'(\lambda_1 x) + \theta_2\lambda_2 u'(\lambda_2 x)\right)(\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)).$$

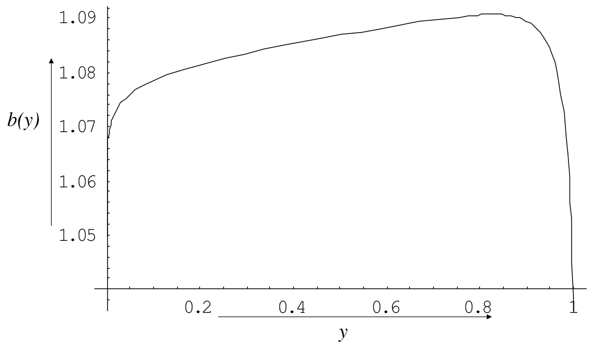

Fig. 2.2. Graph of $b(y)$.

To prove that $\eta_1'(x) \geq (\leq)\ 0$ in x, we have to show that

$$\Psi(\alpha_1, \alpha_2) = \frac{\beta}{x}\frac{\alpha_1 u(\lambda_1 x)v(\lambda_1 x) + \alpha_2 u(\lambda_2 x)v(\lambda_2 x)}{\alpha_1 u(\lambda_1 x) + \alpha_2 u(\lambda_2 x)}$$

is Schur-convex (Schur-concave) in (α_1, α_2) with $v(x)$ as defined earlier. Now,

$$\frac{\partial \Psi}{\partial \alpha_1} - \frac{\partial \Psi}{\partial \alpha_2} \overset{sign}{=} (\alpha_1 + \alpha_2)\left(v(\lambda_1 x) - v(\lambda_2 x)\right) u(\lambda_1 x)u(\lambda_2 x).$$

Thus, if $\boldsymbol{\lambda} \in \mathcal{E}_+$ then $\frac{\partial \Psi}{\partial \alpha_1} - \frac{\partial \Psi}{\partial \lambda_2} \geq 0$. Now, using Lemma 3.1 of Kundu et al. (2016) and noting the fact that $\boldsymbol{\alpha} \in \mathcal{D}_+$, it can be proved that Ψ is Schur-convex in $\boldsymbol{\alpha}$. Again, if $\boldsymbol{\lambda} \in \mathcal{D}_+$ we see that $\frac{\partial \Psi}{\partial \alpha_1} - \frac{\partial \Psi}{\partial \lambda_2} \leq 0$. So, if $\boldsymbol{\alpha} \in \mathcal{D}_+$ then by Lemma 3.1 of Kundu et al. (2016) it can be concluded that Ψ is Schur-concave in $\boldsymbol{\alpha}$. So, if $\boldsymbol{\alpha}, \boldsymbol{\theta} \in \mathcal{D}_+$ and $\boldsymbol{\lambda} \in \mathcal{E}_+$, then $\frac{r_{2:2}(x)}{s_{2:2}(x)}$ is increasing in x and if $\boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_+$, then $\frac{r_{2:2}(x)}{s_{2:2}(x)}$ is decreasing in x. □

The following counterexample shows that Theorem 3.5 does not hold for $n \geq 3$.

Counterexample 3.2. Let $X_i \sim EW(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW(\theta_i, \beta, \lambda_i)$, $i = 1, 2, 3$, where $(\lambda_1, \lambda_2, \lambda_3) = (500, 0.01, 0.001) \in \mathcal{D}_+$, $(\alpha_1, \alpha_2, \alpha_3) = (1, 5, 6) \in \mathcal{E}_+$, $(\theta_1, \theta_2, \theta_3) = (2, 5, 5) \in \mathcal{E}_+$ and $\beta = 0.4$. Clearly, $(\alpha_1, \alpha_2, \alpha_3) \overset{m}{\succeq} (\theta_1, \theta_2, \theta_3)$. By substituting $x = -\ln y$, we see from Fig. 2.2 that $\frac{r_{3:3}(x)}{s_{3:3}(x)} = b(y)$ is not monotone.

Corollary 3.2. *For $i = 1, 2$, let X_i and Y_i be two sets of mutually independent random variables with $X_i \sim EW(\alpha_i, \beta, \lambda_i)$ and $Y_i \sim EW(\theta_i, \beta, \lambda_i)$. Further, suppose that $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\theta}$. Then*

(i) *$X_{2:2} \geq_{lr} Y_{2:2}$ if $\boldsymbol{\alpha}, \boldsymbol{\theta} \in \mathcal{D}_+$ and $\boldsymbol{\lambda} \in \mathcal{E}_+$.*
(ii) *$X_{2:2} \leq_{lr} Y_{2:2}$ if $\boldsymbol{\alpha}, \boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_+$.*

Proof. Using Theorems 3.2 and 3.5 and following similar argument as in the proof of Corollary 3.1, the results follow. □

Although Theorem 3.4 does not hold for $n \geq 3$, the following theorem shows that in case of multiple-outlier model the result is true for any positive integer n.

Theorem 3.6. *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of mutually independent random variables each following the multiple-outlier EW model such that $X_i \sim EW(\alpha, \beta, \lambda)$ and $Y_i \sim EW(\alpha, \beta, \delta)$ for $i = 1, 2, \ldots, n_1$, $X_i \sim EW(\alpha^*, \beta, \lambda^*)$ and $Y_i \sim EW(\alpha^*, \beta, \delta^*)$ for $i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 (=n)$. For $0 < \beta \leq 1$, if*

$$(\underbrace{\lambda, \lambda, \ldots, \lambda}_{n_1}, \underbrace{\lambda^*, \lambda^*, \ldots, \lambda^*}_{n_2}) \overset{m}{\succeq} (\underbrace{\delta, \delta, \ldots, \delta}_{n_1}, \underbrace{\delta^*, \delta^*, \ldots, \delta^*}_{n_2})$$

and $\alpha \leq \alpha^*$, $\lambda \geq \lambda^*$, $\delta \geq \delta^*$, then $X_{n:n} \geq_{lr} Y_{n:n}$.

Proof. Using (3.1) and (3.2) we get,

$$\frac{r_{n:n}(x)}{s_{n:n}(x)} = \frac{\sum_{i=1}^{n} \frac{\alpha_i \beta \lambda_i^\beta x^{\beta-1}}{e^{(\lambda_i x)^\beta} - 1}}{\sum_{i=1}^{n} \frac{\alpha_i \beta \delta_i^\beta x^{\beta-1}}{e^{(\delta_i x)^\beta} - 1}} = \frac{\sum_{i=1}^{n} \alpha_i u(\lambda_i x)}{\sum_{i=1}^{n} \alpha_i u(\delta_i x)} = \eta_2(x)(say),$$

where $\lambda_i = \lambda, \delta_i = \delta, \alpha_i = \alpha$ for $i = 1, 2, \ldots, n_1$ and $\lambda_i = \lambda^*, \delta_i = \delta^*, \alpha_i = \alpha^*$ for $i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 = n$ and $u(x)$ is as defined earlier. So, in view of Theorem 3.1 we need only to prove that $\eta_2(x)$ is increasing in x. Note that

$$\eta_2'(x) \overset{sign}{=} \left(\sum_{i=1}^{n} \alpha_i \lambda_i u'(\lambda_i x)\right)\left(\sum_{i=1}^{n} \alpha_i u(\delta_i x)\right) - \left(\sum_{i=1}^{n} \alpha_i \delta_i u'(\delta_i x)\right)\left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right).$$

Thus, $\eta_2(x)$ is increasing in x, if

$$\Psi(\lambda, x) = \frac{\sum_{i=1}^{n} \alpha_i \lambda_i u'(\lambda_i x)}{\sum_{i=1}^{n} \alpha_i u(\lambda_i x)} = \frac{\beta}{x} \frac{\sum_{i=1}^{n} \alpha_i u(\lambda_i x) v(\lambda_i x)}{\sum_{i=1}^{n} \alpha_i u(\lambda_i x)}$$

is Schur-convex in $\boldsymbol{\lambda}$, where $v(x)$ is as defined earlier. Further,

$$\begin{aligned}\frac{\partial \Psi}{\partial \lambda_i} &= \alpha_i \left[w(\lambda_i x)\left(n_1 \alpha u(\lambda x) + n_2 \alpha^* u(\lambda^* x)\right) + n_1 \alpha u(\lambda x) u'(\lambda_i x)\left(v(\lambda_i x) - v(\lambda x)\right)\right. \\ &\quad \left. + n_2 \alpha^* u(\lambda^* x) u'(\lambda_i x)\left(v(\lambda_i x) - v(\lambda^* x)\right)\right] \cdot \frac{\beta}{x\left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2},\end{aligned}$$

where $w(x)$ is as defined in Theorem 3.4.

Now, three cases may arise:

Case (i) $1 \le i < j \le n_1$. Here $\alpha_i = \alpha_j = \alpha$ and $\lambda_i = \lambda_j = \lambda$, so that

$$\begin{aligned}\frac{\partial \Psi}{\partial \lambda_i} &= \alpha \left[w(\lambda x)\left(n_1 \alpha u(\lambda x) + n_2 \alpha^* u(\lambda^* x)\right) + n_2 \alpha^* u(\lambda^* x) u'(\lambda x)\left(v(\lambda x) - v(\lambda^* x)\right)\right] \cdot \frac{\beta}{x\left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2} \\ &= \frac{\partial \Psi}{\partial \lambda_j},\end{aligned}$$

giving $\frac{\partial \Psi}{\partial \lambda_i} - \frac{\partial \Psi}{\partial \lambda_j} = 0$.

Case (ii) If $n_1 + 1 \le i < j \le n$, *i.e.* if $\alpha_i = \alpha_j = \alpha^*$ and $\lambda_i = \lambda_j = \lambda^*$, then

$$\begin{aligned}\frac{\partial \Psi}{\partial \lambda_i} &= \alpha^* \left[w(\lambda^* x)\left(n_1 \alpha u(\lambda x) + n_2 \alpha^* u(\lambda^* x)\right) + n_1 \alpha u(\lambda x) u'(\lambda^* x)\left(v(\lambda^* x) - v(\lambda x)\right)\right] \cdot \frac{\beta}{x\left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2} \\ &= \frac{\partial \Psi}{\partial \lambda_j},\end{aligned}$$

giving $\frac{\partial \Psi}{\partial \lambda_i} - \frac{\partial \Psi}{\partial \lambda_j} = 0$.

Case (iii) If $1 \le i \le n_1$ and $n_1 + 1 \le j \le n$, then $\alpha_i = \alpha, \lambda_i = \lambda$ and $\alpha_j = \alpha^*, \lambda_j = \lambda^*$. So,

$$\begin{aligned}\frac{\partial \Psi}{\partial \lambda_i} - \frac{\partial \Psi}{\partial \lambda_j} &= \left(n_1 \alpha u(\lambda x) + n_2 \alpha^* u(\lambda^* x)\right)\left(\alpha w(\lambda x) - \alpha^* w(\lambda^* x)\right) \\ &\quad + \left(v(\lambda x) - v(\lambda^* x)\right)\left(n_1 \alpha u(\lambda x) u'(\lambda^* x) + n_2 \alpha^* u(\lambda^* x) u'(\lambda x)\right) \cdot \frac{\beta}{x\left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2}.\end{aligned}$$

Now, if $\lambda \ge \lambda^*$ and $\alpha \le \alpha^*$, we have, for all $x \ge 0$,

$$\alpha w(\lambda x) - \alpha^* w(\lambda^* x) \ge 0.$$

Again, as $v(x)$ and $u(x)$ are decreasing in x, we have $\frac{\partial \Psi}{\partial \lambda_i} - \frac{\partial \Psi}{\partial \lambda_j} \ge 0$ for all x. Hence, by Lemma 3.1 of Kundu et al. (2016), Ψ is Schur-convex.

Theorem 3.7. *For $i = 1, 2, \ldots, n$, let X_i and Y_i be two sets of independent random variables each following the multiple-outlier EW model such that $X_i \sim EW(\alpha, \beta, \lambda)$ and $Y_i \sim EW(\theta, \beta, \lambda)$ for $i = 1, 2, \ldots, n_1$, $X_i \sim EW(\alpha^*, \beta, \lambda^*)$ and $Y_i \sim EW$*

$(\theta^*, \beta, \lambda^*)$ for $i = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 (= n)$. If

$$(\underbrace{\alpha, \alpha, \dots, \alpha}_{n_1}, \underbrace{\alpha^*, \alpha^*, \dots, \alpha^*}_{n_2}) \overset{m}{\succeq} (\underbrace{\theta, \theta, \dots, \theta}_{n_1}, \underbrace{\theta^*, \theta^*, \dots, \theta^*}_{n_2}),$$

and

(i) *$\alpha \le \alpha^*, \theta \le \theta^*, \lambda \ge \lambda^*$, then $X_{n:n} \ge_{lr} Y_{n:n}$;*
(ii) *$\alpha \le \alpha^*, \theta \le \theta^*, \lambda \le \lambda^*$, then $X_{n:n} \le_{lr} Y_{n:n}$.*

Proof. In view of Theorem 3.2 it is sufficient to prove that $\frac{r_{n:n}(x)}{s_{n:n}(x)}$ is increasing in x under condition (i) and $\frac{r_{n:n}(x)}{s_{n:n}(x)}$ is decreasing in x under condition (ii). Using similar argument as before it can be shown that

$$\frac{r_{n:n}(x)}{s_{n:n}(x)} = \frac{\sum_{i=1}^{n} \alpha_i u(\lambda_i x)}{\sum_{i=1}^{n} \theta_i u(\lambda_i x)} = \eta_3(x) (say),$$

with $\lambda_i = \lambda, \theta_i = \theta$ for $i = 1, 2, \dots, n_1$ and $\alpha_i = \alpha^*, \lambda_i = \lambda^*, \theta_i = \theta^*$ for $i = n_1 + 1, n_1 + 2, \dots, n_1 + n_2 = n$. Now,

$$\eta_3'(x) \overset{sign}{=} \left(\sum_{i=1}^{n} \alpha_i \lambda_i u'(\lambda_i x)\right)\left(\sum_{i=1}^{n} \theta_i u(\lambda_i x)\right) - \left(\sum_{i=1}^{n} \theta_i \lambda_i u'(\lambda_i x)\right)\left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right).$$

So, $\eta_3(x)$ is increasing (decreasing) in x if

$$\Psi(\alpha, x) = \frac{\sum_{i=1}^{n} \alpha_i \lambda_i u'(\lambda_i x)}{\sum_{i=1}^{n} \alpha_i u(\lambda_i x)} = \frac{\beta}{x} \frac{\sum_{i=1}^{n} \alpha_i u(\lambda_i x) v(\lambda_i x)}{\sum_{i=1}^{n} \alpha_i u(\lambda_i x)}$$

is Schur-convex (Schur-concave) in $\boldsymbol{\alpha}$. Now,

$$\frac{\partial \Psi}{\partial \alpha_i} = n_1 \alpha u(\lambda x) u(\lambda_i x) \left(v(\lambda_i x) - v(\lambda x)\right) + n_2 \alpha^* u(\lambda^* x) u(\lambda_i x) \left(v(\lambda_i x) - v(\lambda^* x)\right) \cdot \frac{\beta}{x \left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2}.$$

Here also, as before, three cases may arise:

Case (i) $1 \le i < j \le n_1$. Now,

$$\frac{\partial \Psi}{\partial \alpha_i} = n_2 \alpha^* u(\lambda^* x) u(\lambda x) \left(v(\lambda x) - v(\lambda^* x)\right) \cdot \frac{\beta}{x \left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2}$$
$$= \frac{\partial \Psi}{\partial \alpha_j},$$

implying $\frac{\partial \Psi}{\partial \alpha_i} - \frac{\partial \Psi}{\partial \alpha_j} = 0$.

Case (ii) $n_1 + 1 \le i < j \le n$. Here,

$$\frac{\partial \Psi}{\partial \alpha_i} = n_1 \alpha u(\lambda x) u'(\lambda^* x) \left(v(\lambda^* x) - v(\lambda x)\right) \cdot \frac{\beta}{x \left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2}$$
$$= \frac{\partial \Psi}{\partial \alpha_j},$$

giving that $\frac{\partial \Psi}{\partial \alpha_i} - \frac{\partial \Psi}{\partial \alpha_j} = 0$.

Case (iii) $1 \le i \le n_1$ and $n_1 + 1 \le j \le n$. Then,

$$\frac{\partial \Psi}{\partial \alpha_i} - \frac{\partial \Psi}{\partial \alpha_j} = (n_1 \alpha + n_2 \alpha *) u(\lambda x) u(\lambda^* x) \left(v(\lambda x) - v(\lambda^* x)\right) \cdot \frac{\beta}{x \left(\sum_{i=1}^{n} \alpha_i u(\lambda_i x)\right)^2}.$$

Now, if $\lambda \geq \lambda^*$ then $(v(\lambda x) - v(\lambda^* x)) \leq 0$, which gives that

$$\frac{\partial \Psi}{\partial \alpha_i} - \frac{\partial \Psi}{\partial \alpha_j} \leq 0.$$

So, if $\alpha \leq \alpha^*$, then by Lemma 3.3 of Kundu et al. (2016), Ψ is Schur-convex.

On the other hand, if $\lambda \leq \lambda^*$ then $(v(\lambda x) - v(\lambda^* x)) \geq 0$ which implies that

$$\frac{\partial \Psi}{\partial \alpha_i} - \frac{\partial \Psi}{\partial \alpha_j} \geq 0.$$

Hence, if $\alpha \leq \alpha^*$, then by Lemma 3.3 of Kundu et al. (2016), Ψ is Schur-concave. □

4. Concluding remarks

In this paper, we compare the reversed hazard rate functions of the largest order statistic arising from independent heterogeneous EW distributions when the scale parameters or the shape parameters are majorized. It is also shown that if the vectors of scale or shape parameters of the underlying distributions are in majorization order, then likelihood ratio ordering exists between the largest order statistic of both the two-component systems. Further, we prove that, in the multiple-outlier EW model, if one set of scale parameters (shape parameters) majorizes another, a parallel system formed by the former will dominate that formed by the latter in the likelihood ratio order. All the results of this paper related to EW random variables improve similar results related to GE, Weibull, generalized Rayleigh and exponential random variables. It is not very difficult to show that the results proved for the set of parameters $\boldsymbol{\alpha} \in \mathcal{E}_+$ and $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$ will also be true for the set $\boldsymbol{\alpha} \in \mathcal{D}_+$ and $\boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{E}_+$. Further, the results which hold for the set $\boldsymbol{\alpha}, \boldsymbol{\lambda} \in \mathcal{D}_+$ and $\boldsymbol{\delta} \in \mathcal{E}_+$ will also hold for $\boldsymbol{\alpha}, \boldsymbol{\lambda} \in \mathcal{E}_+$ and $\boldsymbol{\delta} \in \mathcal{D}_+$. Similarly, the results which hold for $\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{D}_+$ will also hold for $\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{\delta} \in \mathcal{E}_+$.

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